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# A rigorous proof of a conjecture for the one-dimensional perturbed Gelfand problem from combustion theory

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## 1. Introduction

We study the global bifurcation curves and exact multiplicity of positive solutions for the two-point boundary value problem

$$\begin{cases} u''(x) + \lambda \exp\left(\frac{au}{a+u}\right) = 0, & -1 < x < 1, \\ u(-1) = u(1) = 0, \end{cases} \quad (1.1)$$

which is the one-dimensional case of a problem arising in the study of standard models of ignition in a context of thermal combustion, cf. [1, 23]. In (1.1),  $\lambda > 0$  is the Frank-Kamenetskii parameter or ignition parameter,  $a > 0$  is the activation energy parameter,  $u$  is the dimensionless temperature of the medium, and the reaction term

$$f_a(u) \equiv \exp\left(\frac{au}{a+u}\right)$$

is the temperature dependence obeying the simple Arrhenius reaction-rate law in irreversible chemical reaction kinetics, see, e.g. Boddington et al. [2]. Notice that nonlinearity  $f_a \in C^\infty[0, \infty)$  satisfies  $f_a(u)$ ,  $f'_a(u) > 0$  for  $u \geq 0$  and  $a > 0$ . In addition,  $f''_a(u)$  is negative (concave) for  $0 < a \leq 2$ , and  $f''_a(u)$  is positive and then negative (convex-concave) for  $a > 2$ .

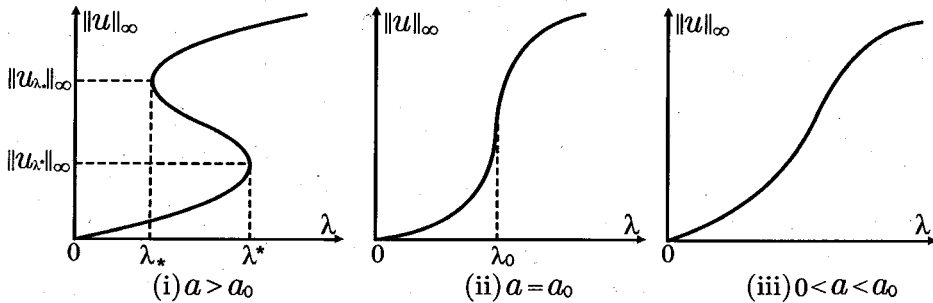


Figure 1.1: The global bifurcation of bifurcation curves  $S_a$  with varying  $a > 0$ .

For any  $a > 0$ , on the  $(\lambda, \|u\|_\infty)$ -plane, we study the shape and structure of bifurcation curves  $S_a$  of positive solutions of (1.1), defined by

$$S_a \equiv \{(\lambda, \|u_\lambda\|_\infty) : \lambda > 0 \text{ and } u_\lambda \text{ is a positive solution of (1.1)}\}.$$

We say that, on the  $(\lambda, \|u_\lambda\|_\infty)$ -plane, the bifurcation curve  $S_a$  is S-shaped if  $S_a$  has *exactly two* turning points at some points  $(\lambda^*, \|u_{\lambda^*}\|_\infty)$  and  $(\lambda_*, \|u_{\lambda_*}\|_\infty)$  where  $\lambda_* < \lambda^*$  are two positive numbers such that

- (i)  $\|u_{\lambda^*}\|_\infty < \|u_{\lambda_*}\|_\infty$ ,
- (ii) at  $(\lambda^*, \|u_{\lambda^*}\|_\infty)$  the bifurcation curve  $S_a$  turns to the left,
- (iii) at  $(\lambda_*, \|u_{\lambda_*}\|_\infty)$  the bifurcation curve  $S_a$  turns to the right.

See Figure 1.1(i).

It is important to notice that, substituting  $a = 1/\varepsilon$  ( $\varepsilon$  is the reciprocal activation energy parameter) into (1.1), we obviously obtain

$$\begin{cases} u''(x) + \lambda \exp\left(\frac{u}{1+\varepsilon u}\right) = 0, & -1 < x < 1, \\ u(-1) = u(1) = 0. \end{cases} \quad (1.2)$$

This problem (1.2) is the famous one-dimensional perturbed Gelfand problem, cf. [1, 6, 9]. It has been a long-standing conjecture ([5, 6, 15, 16, 20, 21]) about the shapes of evolutionary bifurcation curves and the exact multiplicity of positive solutions of (1.2) with varying  $\varepsilon > 0$ . This problem is obviously equivalent to study the shapes of evolutionary bifurcation curves and the exact multiplicity of positive solutions of (1.1) with varying  $a > 0$ . The conjecture for one-dimensional perturbed Gelfand problem (1.2) is stated in the form of (1.1) as follows.

**Conjecture 1.1.** *Consider (1.1) with varying  $a > 0$ . There exists a critical bifurcation value  $a_0 > 4$  such that the following assertions (i)–(iii) hold:*

- (i) (See Figure 1.1(i).) *For  $a > a_0$ , the bifurcation curve  $S_a$  is S-shaped on the  $(\lambda, \|u\|_\infty)$ -plane. More precisely, there exist two positive numbers  $\lambda_* < \lambda^*$  such that (1.1) has exactly three positive solutions for  $\lambda_* < \lambda < \lambda^*$ , exactly two positive solutions for  $\lambda = \lambda_*$  and  $\lambda = \lambda^*$ , and exactly one positive solution for  $0 < \lambda < \lambda_*$  and  $\lambda > \lambda^*$ . Furthermore, all positive solutions  $u_\lambda$  are nondegenerate except that  $u_{\lambda_*}$  and  $u_{\lambda^*}$  are degenerate.*
- (ii) (See Figure 1.1(ii).) *For  $a = a_0$ , the bifurcation curve  $S_a$  of (1.1) is monotone increasing on the  $(\lambda, \|u\|_\infty)$ -plane. More precisely, for all  $\lambda > 0$ , (1.1) has exactly one positive solution  $u_\lambda$ . Furthermore, all positive solutions  $u_\lambda$  are nondegenerate except that  $u_{\lambda_0}$  is degenerate for some  $\lambda_0 > 0$ .*
- (iii) (See Figure 1.1(iii).) *For  $0 < a < a_0$ , the bifurcation curve  $S_a$  of (1.1) is monotone increasing on the  $(\lambda, \|u\|_\infty)$ -plane. More precisely, for all  $\lambda > 0$ , (1.1) has exactly one positive solution  $u_\lambda$ . Furthermore, all positive solutions  $u_\lambda$  are nondegenerate.*

Note that Korman, Li and Ouyang [16] gave a computer-assisted proof of this conjecture. Many researchers devoted to solve this conjecture since the 1980s. For  $0 < a \leq 4$ , it is easy to prove that the bifurcation curve  $S_a$  is monotone increasing and all positive solutions of (1.1) are nondegenerate, and hence  $a_0 > 4$  under Conjecture 1.1, see e.g. [3]. In 1981, using quadratures, Brown et. al [3] showed that, for  $a > \check{a}_1 \approx 4.25$  for some  $\check{a}_1$ , the bifurcation curve  $S_a$  is S-like shaped (i.e.,  $S_a$  has at least two turning points). In 1985, using quadratures, Hastings and McLeod [8] proved that the bifurcation curve  $S_a$  is S-shaped for sufficiently large  $a$ . In 1994, again using quadratures, Wang [22] proved that the bifurcation curve  $S_a$  is S-shaped for  $a \geq \check{a}_2 \approx 4.4967$  for some  $\check{a}_2$ , and hence

$a_0 < \check{a}_2 \approx 4.4967$  under Conjecture 1.1. In 1999, Korman and Li [15] reduced the upper bound  $\check{a}_2$  of  $a_0$  to  $\check{a}_3 \approx 4.35$  for some  $\check{a}_3$ . They used tools from bifurcation theory, particularly the Crandall-Rabinowitz bifurcation theorem [4], and used quadratures. In 2011, again using quadratures, Hung and Wang [12] proved that the bifurcation curve  $S_a$  is S-shaped for  $a \geq a^*$  where

$$a^* \equiv \inf \left\{ a > 4 : \int_0^{\frac{a(a-2)}{2}} [u f_a(u) - u^2 f'_a(u)] du < 0 \right\} \approx 4.166, \quad (1.3)$$

and hence  $a_0 < a^* \approx 4.166$  under Conjecture 1.1. Very recently in 2015, using quadratures again together with Sturm's theorem, Huang and Wang [10] proved that the bifurcation curve  $S_a$  is S-shaped for  $a \geq \tilde{a}$  where

$$\tilde{a} \equiv \inf \left\{ a > 4 : \int_0^{\frac{a(a-2) + a\sqrt{a(a-4)}}{2}} [u f_a(u) - u^2 f'_a(u)] du < 0 \right\} \approx 4.107, \quad (1.4)$$

and hence  $a_0 < \tilde{a} \approx 4.107$  under Conjecture 1.1. So by above, we have the following theorem.

**Theorem 1.2.** *Consider (1.1) with  $a > 0$ . Then the following assertions (i) and (ii) hold:*

- (i) *For  $0 < a \leq 4$ , the bifurcation curve  $S_a$  is monotone increasing on the  $(\lambda, \|u\|_\infty)$ -plane. Furthermore, all positive solutions  $u_\lambda$  are nondegenerate.*
- (ii) *For  $a \geq \tilde{a} \approx 4.107$ , the bifurcation curve  $S_a$  is S-shaped on the  $(\lambda, \|u\|_\infty)$ -plane.*

Write  $\exp\left(\frac{au}{a+u}\right) = \exp\left(\frac{u}{1+\varepsilon u}\right)$  with  $\varepsilon = 1/a$ . Thus, for fixed  $u$ ,  $\exp\left(\frac{u}{1+\varepsilon u}\right) \rightarrow \exp(u)$  as  $\varepsilon \rightarrow 0^+$  (i.e.,  $a \rightarrow \infty$ ). In that case problem (1.1) and problem (1.2) reduce to the one-dimensional Gelfand problem (or called Liouville-Gelfand problem)

$$\begin{cases} u''(x) + \lambda \exp(u) = 0, & -1 < x < 1, \\ u(-1) = u(1) = 0. \end{cases} \quad (1.5)$$

In 1853, Liouville [18] first studied (1.5) and found an explicit solution. In 1959, Gelfand [7] observed that problem (1.5) can be solved by integration exactly, with positive solution

$$u_\lambda(x) = \alpha + \ln \left( \operatorname{sech}^2 \left( \frac{\sqrt{2\lambda}}{2} x e^{\alpha/2} \right) \right),$$

where  $\alpha \equiv \|u_\lambda\|_\infty = u_\lambda(0)$ . This enabled him to deduce that (1.5) has either two, one, or zero solutions, depending on  $\lambda$ , see [9, p. 208] and [1, p. 34].

Define

$$S_\infty \equiv \{(\lambda, \|u_\lambda\|_\infty) : \lambda > 0 \text{ and } u_\lambda \text{ is a positive solution of (1.5)}\}.$$

Then by the quadrature method (time-map method) and the fact that the nonlinearity  $\exp(u)$  has an elementary antiderivative  $\exp(u)$ , one finds that

$$\lambda = \lambda(\alpha) = \left[ \frac{1}{\sqrt{2}} \int_0^\alpha \frac{1}{\sqrt{e^\alpha - e^u}} du \right]^2 = \frac{1}{2e^\alpha} \left[ \ln \left( 2e^\alpha + 2\sqrt{e^\alpha(e^\alpha - 1)} - 1 \right) \right]^2 \quad \text{for } \alpha > 0 \quad (1.6)$$

after some simple computation, see e.g. [14, Eq. (5)]. It is easy to show that  $\lim_{\alpha \rightarrow 0^+} \lambda(\alpha) = \lim_{\alpha \rightarrow \infty} \lambda(\alpha) = 0$  and  $\lambda(\alpha)$  has exact one critical (maximum) value

$$\lambda_\infty \equiv \max_{\alpha \in (0, \infty)} \frac{1}{2e^\alpha} \left[ \ln \left( 2e^\alpha + 2\sqrt{e^\alpha(e^\alpha - 1)} - 1 \right) \right]^2 \approx 0.878 \quad (1.7)$$

at some critical point

$$\alpha_\infty = \ln \left( \frac{2 + \lambda_\infty}{\lambda_\infty} \right) \approx 1.187 \quad (1.8)$$

after some simple computation; we omit the proofs. Thus the bifurcation curve  $S_\infty$  is a  $\supset$ -shaped curve on the  $(\lambda, \|u\|_\infty)$ -plane and the next theorem follows.

**Theorem 1.3.** *Consider (1.5). There exist a critical (maximal) value  $\lambda_\infty \approx 0.878$  and a critical point  $\alpha_\infty = \ln \left( \frac{2 + \lambda_\infty}{\lambda_\infty} \right) \approx 1.187$  for  $\lambda(\alpha)$  in (1.6) such that the bifurcation curve  $S_\infty$  is a  $\supset$ -shaped curve on the  $(\lambda, \|u\|_\infty)$ -plane and satisfies (1.6)–(1.8). More precisely, (1.5) has exactly two positive solutions for  $0 < \lambda < \lambda_\infty$ , exactly one positive solution for  $\lambda = \lambda_\infty$ , and no positive solution for  $\lambda > \lambda_\infty$ . In addition,  $\|u_{\lambda_\infty}\|_\infty = \alpha_\infty$ .*

## 2. Main result

**Theorem 2.1.** *Consider (1.1) with varying  $a > 0$ . There exists a critical bifurcation value  $a_0 \approx 4.069$  satisfying  $4 < a_0 < \tilde{a} \approx 4.107$  such that the following assertions (i)–(iii) hold:*

- (i) (See Figure 1.1(i).) *For  $a > a_0$ , the bifurcation curve  $S_a$  is S-shaped on the  $(\lambda, \|u\|_\infty)$ -plane. More precisely, there exist two positive numbers  $\lambda_* < \lambda^*$  such that (1.1) has:*
  - (a) *exactly three positive solutions  $w_\lambda, u_\lambda, v_\lambda$  with  $w_\lambda < u_\lambda < v_\lambda$  for  $\lambda_* < \lambda < \lambda^*$ ,*
  - (b) *exactly two positive solutions  $w_{\lambda_*}, u_{\lambda_*}$  with  $w_{\lambda_*} < u_{\lambda_*}$  for  $\lambda = \lambda_*$ , and exactly two positive solutions  $u_{\lambda^*}, v_{\lambda^*}$  with  $u_{\lambda^*} < v_{\lambda^*}$  for  $\lambda = \lambda^*$ ,*
  - (c) *exactly one positive solution  $w_\lambda$  for  $0 < \lambda < \lambda_*$ , and exactly one positive solution  $v_\lambda$  for  $\lambda > \lambda^*$ .*

Furthermore,

- (d)  $\lim_{\lambda \rightarrow 0^+} \|w_\lambda\|_\infty = 0$  and  $\lim_{\lambda \rightarrow \infty} \|v_\lambda\|_\infty = \infty$ .
- (e) *All positive solutions  $u_\lambda$  are nondegenerate except that  $u_{\lambda_*}$  and  $u_{\lambda^*}$  are degenerate.*
- (f)  $\|u_{\lambda^*}\|_\infty < \|u_{\lambda_*}\|_\infty$ ,  $\lim_{a \rightarrow \infty} \|u_{\lambda_*}\|_\infty = \infty$  and  $\lim_{a \rightarrow \infty} \|u_{\lambda^*}\|_\infty = \alpha_\infty \approx 1.187$ .
- (ii) (See Figure 1.1(ii).) *If  $a = a_0$ , then the bifurcation curve  $S_{a_0}$  is monotone increasing on the  $(\lambda, \|u\|_\infty)$ -plane. More precisely, for all  $\lambda > 0$ , (1.1) has exactly one positive solution  $u_\lambda$  satisfying  $\lim_{\lambda \rightarrow 0^+} \|u_\lambda\|_\infty = 0$  and  $\lim_{\lambda \rightarrow \infty} \|u_\lambda\|_\infty = \infty$ . Furthermore, all positive solutions  $u_\lambda$  are nondegenerate except that  $u_{\lambda_0}$  is a degenerate solution for some  $\lambda = \lambda_0 > 0$ .*
- (iii) (See Figure 1.1(iii).) *If  $0 < a < a_0$ , then the bifurcation curve  $S_a$  is monotone increasing on the  $(\lambda, \|u\|_\infty)$ -plane. More precisely, for all  $\lambda > 0$ , (1.1) has exactly one positive solution  $u_\lambda$  satisfying  $\lim_{\lambda \rightarrow 0^+} \|u_\lambda\|_\infty = 0$  and  $\lim_{\lambda \rightarrow \infty} \|u_\lambda\|_\infty = \infty$ . Furthermore, all positive solutions  $u_\lambda$  are nondegenerate.*

## 3. Lemmas

To prove Theorem 2.1, we develop some new time-map techniques and apply Sturm's theorem stated in [11]. The time-map formula which we apply to study (1.1) takes the form as follows:

$$\sqrt{\lambda} = \frac{1}{\sqrt{2}} \int_0^\alpha [F_a(\alpha) - F_a(u)]^{-1/2} du \equiv T_a(\alpha) \quad \text{for } 0 < \alpha < \infty, \quad (3.1)$$

where  $F_a(u) \equiv \int_0^u f_a(t)dt$ . So positive solutions  $u$  of (1.1) correspond to

$$\|u\|_\infty = \alpha \text{ and } T_a(\alpha) = \sqrt{\lambda}.$$

Thus, studying the exact number of positive solutions of (1.1) is equivalent to studying the shape of the time map  $T_a(\alpha)$  on  $(0, \infty)$ , cf. [10]. In addition, proving that the bifurcation curve  $S_a$  is S-shaped on the  $(\lambda, \|u\|_\infty)$ -plane is equivalent to proving that  $T_a(\alpha)$  has *exactly two* critical points, a local maximum and a local minimum, on  $(0, \infty)$ . We recall that a positive solution  $u_\lambda$  of (1.1) is *degenerate* if  $T'_a(\|u_\lambda\|_\infty) = 0$  and is *nondegenerate* if  $T'_a(\|u_\lambda\|_\infty) \neq 0$ .

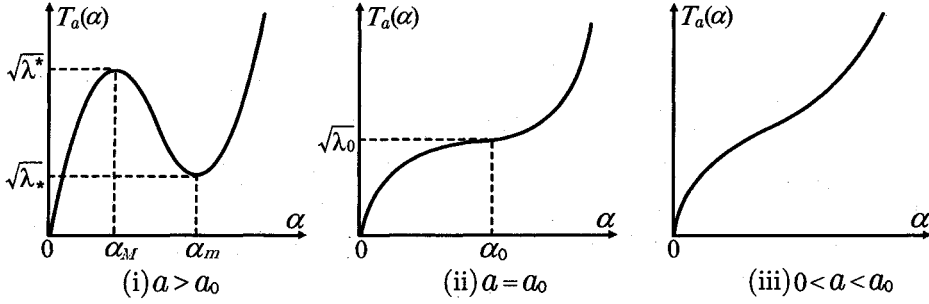


Figure 3.1: Graphs of  $T_a(\alpha) (= \sqrt{\lambda})$  on  $(0, \infty)$  with varying  $a > 0$ , cf. Figure 1.1.

By Theorem 1.2, we see that  $T_a(\alpha)$  is strictly increasing and has no critical points on  $(0, \infty)$  for  $0 < a \leq 4$ , and  $T_a(\alpha)$  has exactly two critical points, a local maximum and a local minimum, on  $(0, \infty)$  for  $a \geq \tilde{a}$ . So by above, to prove Theorem 2.1(i), (ii) and (iii) which solves Conjecture 1.1, it is sufficient to prove that there exists a number  $a_0 \approx 4.069$  satisfying  $4 < a_0 < \tilde{a} \approx 4.107$  such that the following parts (M1), (M2) and (M3) hold, respectively:

- (M1) (See Figure 3.1(i).) For  $a > a_0$ , on  $(0, \infty)$ ,  $T_a(\alpha)$  has *exactly two* critical points, a local maximum at some  $\alpha_M(a)$  and a local minimum at some  $\alpha_m(a)$  ( $> \alpha_M(a)$ ), satisfying  $\lambda^* = T_a^2(\alpha_M(a))$ ,  $\lambda_* = T_a^2(\alpha_m(a))$ . In addition,  $\lim_{\alpha \rightarrow 0^+} T_a(\alpha) = 0$ ,  $\lim_{\alpha \rightarrow \infty} T_a(\alpha) = \infty$ ,  $\lim_{a \rightarrow \infty} \alpha_m(a) = \alpha_\infty$ , and  $\lim_{a \rightarrow \infty} \alpha_M(a) = \infty$  where  $\alpha_\infty$  is defined in (1.8).
- (M2) (See Figure 3.1(ii).) For  $a = a_0$ ,  $T_{a_0}(\alpha)$  is a strictly increasing function on  $(0, \infty)$  and has *exactly one* critical point at some  $\alpha_0$  on  $(0, \infty)$ . Moreover,  $T'_{a_0}(\alpha_0) = 0$  and  $T'_{a_0}(\alpha) > 0$  for  $\alpha \in (0, \infty) \setminus \{\alpha_0\}$ . In addition,  $\lim_{\alpha \rightarrow 0^+} T_{a_0}(\alpha) = 0$  and  $\lim_{\alpha \rightarrow \infty} T_{a_0}(\alpha) = \infty$ ,
- (M3) (See Figure 3.1(iii).) For  $0 < a < a_0$ ,  $T_a(\alpha)$  is a strictly increasing function and has no critical points on  $(0, \infty)$ . Moreover,  $T'_a(\alpha) > 0$  on  $(0, \infty)$ . In addition,  $\lim_{\alpha \rightarrow 0^+} T_a(\alpha) = 0$  and  $\lim_{\alpha \rightarrow \infty} T_a(\alpha) = \infty$ .

To prove parts (M1)–(M3), we need the following Lemmas 3.1–3.3.

**Lemma 3.1.** Consider (1.1) with  $a > 0$ . Then  $\lim_{\alpha \rightarrow 0^+} T_a(\alpha) = 0$  and  $\lim_{\alpha \rightarrow \infty} T_a(\alpha) = \infty$ .

**Lemma 3.2.** Consider (1.1) with  $a > 0$ . The set  $\Omega$  defined by

$$\Omega = \left\{ a > 0 : T_a(\alpha) \text{ has exactly two critical points } \alpha_M(a) < \alpha_m(a), \right. \\ \left. \text{a local maximum and a local minimum, on } (0, \infty) \right\}$$

is nonempty, open and connected. Moreover,  $\Omega = (a_0, \infty)$  for some number  $a_0$  ( $\approx 4.069$ )  $\in (4, \tilde{a})$  where  $\tilde{a}$  is defined in (1.4).

**Lemma 3.3.** Consider (1.1) with  $4 < a \leq a_0$ . The following assertions (i)–(ii) hold:

- (i) For  $4 < a < a_0$ ,  $T'_a(\alpha) > 0$  for  $\alpha > 0$ .
- (ii) For  $a = a_0$ , there exists  $\alpha_0 \in (\gamma(a_0), \kappa(a_0))$  such that  $T'_{a_0}(\alpha_0) = 0$  and  $T'_{a_0}(\alpha) > 0$  for  $\alpha > 0$  and  $\alpha \neq \alpha_0$ , where  $\kappa(a_0)$  is defined in Lemma 3.5 stated below. Moreover,  $4 < a_0 < \tilde{a} \approx 4.107$ .

First of all, Lemma 3.1 follows easily from [17, Theorems 2.6 and 2.9]. Before proving Lemmas 3.2 and 3.3, we need to investigate some properties of  $T_a(\alpha)$  on  $(0, \infty)$ . In fact, we apply the next Lemmas 3.5–3.7 and 3.12 to prove Lemma 3.2, and apply Lemma 3.2 and the next Lemmas 3.5–3.7, 3.11, 3.14 and 3.16 to prove Lemma 3.3.

Next, we divide this section into three subsections.

### 3.1. Basic functions estimates

We first compute and obtain that, for  $u > 0$ ,

$$f'_a(u) = \frac{1}{(a+u)^2} a^2 f_a(u) > 0, \quad (3.2)$$

$$f''_a(u) = -\frac{2a^2 [u - a(a-2)/2]}{(a+u)^4} \exp\left(\frac{au}{a+u}\right) \begin{cases} > 0 & \text{for } u < \gamma(a), \\ = 0 & \text{for } u = \gamma(a) \equiv \frac{a(a-2)}{2}, \\ < 0 & \text{for } u > \gamma(a). \end{cases} \quad (3.3)$$

For the sake of convenience, we let  $\gamma = \gamma(a)$  for  $a > 2$ . We let

$$\theta_a(u) \equiv 2F_a(u) - u f_a(u) = 2 \int_0^u f_a(t) dt - u f_a(u). \quad (3.4)$$

Lemma 3.4(ii) follows easily from [10, Lemma 2.1].

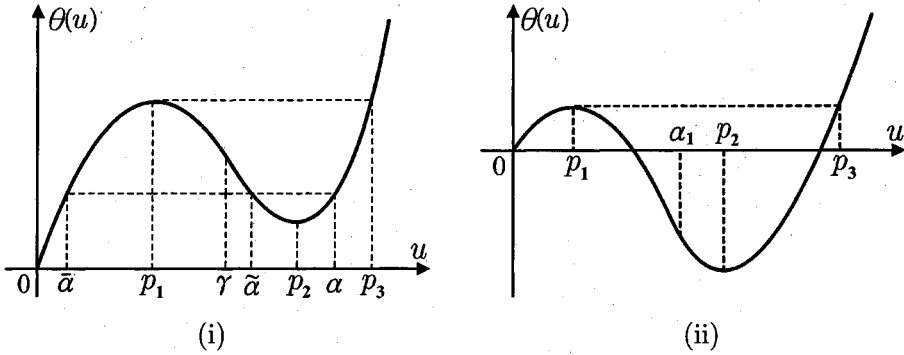


Figure 3.2: Graphs of  $\theta_a(u)$  on  $[0, \infty)$ . (i)  $\theta_a(p_2(a)) \geq 0$ . (ii)  $\theta_a(p_2(a)) < 0$ .

**Lemma 3.4.** Consider (1.1) with  $a > 4$ . Define positive numbers

$$p_1(a) \equiv \frac{a(a-2) - a\sqrt{a(a-4)}}{2} < p_2(a) \equiv \frac{a(a-2) + a\sqrt{a(a-4)}}{2}. \quad (3.5)$$

Then

$$0 < p_1(a) < \gamma(a) = \frac{a(a-2)}{2} < p_2(a), \quad (3.6)$$

and the following assertions (i)–(ii) hold:

(i) (See Figure 3.2.)  $\theta_a(0) = 0$ ,  $\lim_{u \rightarrow \infty} \theta_a(u) = \infty$ , and

$$\theta'_a(u) \begin{cases} > 0 & \text{for } u \in (0, p_1(a)) \cup (p_2(a), \infty), \\ = 0 & \text{for } u \in \{p_1(a), p_2(a)\}, \\ < 0 & \text{for } u \in (p_1(a), p_2(a)). \end{cases} \quad (3.7)$$

Furthermore, there exists a unique number  $p_3(a) > p_2(a)$  such that  $\theta_a(p_3(a)) = \theta_a(p_1(a))$ . In addition,  $p_3(a) < 12$  for  $4 < a \leq \tilde{a} \approx 4.107$ .

(ii) (See Figure 3.2(i).) For  $4 < a \leq \frac{417}{100}$  and  $\alpha \in [\gamma(a), p_3(a))$ ,  $\theta_a(u) > 0$  for  $u > 0$  and there exist two numbers  $\tilde{\alpha} \in (0, p_1(a))$  and  $\tilde{\alpha} \in (p_1(a), p_2(a)]$  such that  $\theta_a(\tilde{\alpha}) = \theta_a(\tilde{\alpha}) = \theta_a(\alpha)$ . (Notes that we choose  $\tilde{\alpha} = \alpha$  if  $\alpha \in [\gamma(a), p_2(a)]$ .)

**Proof of Lemma 3.4.** Since Lemma 3.4(ii) follows easily from [10, Lemma 2.1], it is sufficient to prove Lemma 3.4(i). It is easy to see that (3.6) holds for  $a > 4$ . Secondly, we compute and observe that

$$\begin{aligned} \theta'_a(u) &= f_a(u) - u f'_a(u) \\ &= \frac{[u^2 - a(a-2)u + a^2]}{(a+u)^2} f_a(u) \begin{cases} > 0 & \text{for } u \in (0, p_1(a)) \cup (p_2(a), \infty), \\ = 0 & \text{for } u \in \{p_1(a), p_2(a)\}, \\ < 0 & \text{for } u \in (p_1(a), p_2(a)). \end{cases} \end{aligned} \quad (3.8)$$

Then we simply prove that  $p_3(a) < 12$  for  $4 < a \leq \tilde{a}$  in part (i) because the remainder parts follow easily from [3, p. 482, lines 29–30] and [12, p. 228]. Since  $\tilde{a} (\approx 4.107) < 4.108 = \frac{1027}{250}$  by (1.4), it is sufficient to prove that  $p_3(a) < 12$  for  $4 < a < \frac{1027}{250}$ . Clearly,

$$p'_2(a) = \frac{(a-1)\sqrt{a^2-4a} + a(a-3)}{\sqrt{a^2-4a}} > 0 \quad \text{for } a > 4. \quad (3.9)$$

So we see that

$$p_1(a) < p_2(a) < p_2\left(\frac{1027}{250}\right) = \frac{3081\sqrt{3081}}{125000} + \frac{541229}{125000} (\approx 5.698) < 12 \quad \text{for } 4 < a < \frac{1027}{250}. \quad (3.10)$$

We assert that

$$\theta_a(12) - \theta_a(p_3(a)) > 0 \quad \text{for } 4 < a < \frac{1027}{250}. \quad (3.11)$$

So by (3.8),  $p_3(a) < 12$  for  $4 < a < \frac{1027}{250}$ . It implies that  $p_3(a) < 12$  for  $4 < a \leq \tilde{a}$ .

Next, we prove assertion (3.11). We observe that

$$\frac{\alpha^3 f_a(\alpha)}{(a+\alpha)^2} - \frac{u^3 f_a(u)}{(a+u)^2} = \int_u^\alpha \left[ \frac{d}{dt} \frac{t^3 f_a(t)}{(a+t)^2} \right] dt = \int_u^\alpha \frac{[t^2 + a(a+4)t + 3a^2] t^2 f_a(t)}{(a+t)^4} dt. \quad (3.12)$$



Since  $t^2 - 16t - 16 < 0$  for  $0 \leq t \leq 12$ , and by Lemma 3.4(i), (3.10) and (3.12), we compute and obtain that, for  $4 < a < \frac{1027}{250}$ ,

$$\begin{aligned}
 \frac{\partial}{\partial a} [\theta_a(12) - \theta_a(p_1(a))] &= \frac{\partial}{\partial a} \theta_a(12) - \frac{\partial}{\partial a} \theta_a(u) \Big|_{u=p_1(a)} \\
 &= 2 \int_{p_1(a)}^{12} \frac{t^2 f_a(t)}{(a+t)^2} dt + \frac{p_1^3(a) f_a(p_1(a))}{[a+p_1(a)]^2} - \frac{(12)^3 f_a(12)}{(a+12)^2} \\
 &= \int_{p_1(a)}^{12} \frac{t^2 f_a(t)}{(a+t)^4} (t^2 - a^2 t - a^2) dt \quad (\text{since } a > 4) \\
 &< \int_{p_1(a)}^{12} \frac{t^2 f_a(t)}{(a+t)^4} (t^2 - 16t - 16) dt < 0.
 \end{aligned} \tag{3.13}$$

So by (3.4), (3.8) and (3.10), we compute and obtain that, for  $4 < a < \frac{1027}{250}$ ,

$$\begin{aligned}
 &\theta_a(12) - \theta_a(p_1(a)) \\
 &= \int_{p_1(a)}^{12} \theta'_a(u) du \\
 &= \int_{p_1(a)}^{12} \frac{[u^2 - a(a-2)u + a^2]}{(a+u)^2} f_a(u) du \\
 &\geq f_a(p_2(a)) \left\{ \int_{p_1(a)}^{p_2(a)} \frac{u^2 - a(a-2)u + a^2}{(a+u)^2} du + \int_{p_2(a)}^{12} \frac{[u^2 - a(a-2)u + a^2]}{(a+u)^2} du \right\} \\
 &= f_a(p_2(a)) \int_{p_1(a)}^{12} \frac{u^2 - a(a-2)u + a^2}{(a+u)^2} du = \frac{f_a(p_2(a))}{(a+12)[a+p_1(a)]} \Phi_a,
 \end{aligned} \tag{3.14}$$

where

$$\begin{aligned}
 \Phi_a &\equiv [12 - p_1(a)] \{a^3 + a^2 + [12 + p_1(a)]a + 12p_1(a)\} \\
 &\quad + a^2(a+12)[a+p_1(a)] \ln \left( \frac{a+p_1(a)}{a+12} \right).
 \end{aligned}$$

Since  $\Phi_{\frac{1027}{250}} (\approx 73.2) > 0$ , and by (3.13) and (3.14), we see that, for  $4 < a < \frac{1027}{250}$ ,

$$\begin{aligned}
 \theta_a(12) - \theta_a(p_3(a)) &= \theta_a(12) - \theta_a(p_1(a)) \geq \theta_{\frac{1027}{250}}(12) - \theta_{\frac{1027}{250}} \left( p_1 \left( \frac{1027}{250} \right) \right) \\
 &\geq \frac{f_{\frac{1027}{250}} \left( p_2 \left( \frac{1027}{250} \right) \right)}{\left( \frac{1027}{250} + 12 \right) \left[ \frac{1027}{250} + p_1 \left( \frac{1027}{250} \right) \right]} \Phi_{\frac{1027}{250}} > 0.
 \end{aligned}$$

So assertion (3.11) holds. The proof of Lemma 3.4 is complete. ■

**Lemma 3.5.** Consider (1.1) with  $a > 4$ . Then the following assertions (i)–(ii) hold:

(i) There exists a continuous function  $\kappa(a) \in (\gamma, \infty)$  of  $a$  on  $(4, \infty)$  such that

$$G_a(u) \equiv - \int_0^u t^2 \theta''_a(t) dt \begin{cases} > 0 & \text{for } 0 < u < \kappa(a), \\ = 0 & \text{for } u = \kappa(a), \\ < 0 & \text{for } u > \kappa(a). \end{cases} \tag{3.15}$$

Furthermore,  $\kappa(a)$  is a strictly increasing function of  $a$  on  $(4, \frac{417}{100}]$  and  $\kappa(a) < 8$  for  $4 < a \leq \frac{417}{100}$ .

(ii) There exists a strictly decreasing, continuous function  $\rho(a) \in (0, \kappa(a)]$  of  $a \geq \tilde{a}$  such that

$$H_a(u) \equiv \int_0^u t\theta'_a(t)dt \begin{cases} > 0 & \text{for } 0 < u < \rho(a), \\ = 0 & \text{for } u = \rho(a), \\ < 0 & \text{for } \rho(a) < u < p_2(a). \end{cases} \quad (3.16)$$

Furthermore,

$$\gamma < \rho(a) = \kappa(a) = p_2(a) \quad \text{for } a = \tilde{a}, \quad (3.17)$$

$$\gamma < \rho(a) < \kappa(a) < p_2(a) \quad \text{for } \tilde{a} < a < a^*, \quad (3.18)$$

$$\gamma = \rho(a) < \kappa(a) < p_2(a) \quad \text{for } a = a^*, \quad (3.19)$$

$$p_1(a) < \rho(a) < \gamma < \kappa(a) < p_2(a) \quad \text{for } a > a^*. \quad (3.20)$$

**Proof of Lemma 3.5.** We divide this proof into the next Steps 1-2.

**Step1.** We prove assertion (i). Clearly,  $G_a(0) = 0$ . Since  $\lim_{u \rightarrow \infty} f_a(u) = \exp(a)$  and by (3.3), we compute and find that, for  $a > 4$ ,

$$\begin{aligned} G'_a(u) &= -u^2\theta''_a(u) = u^3f''_a(u) \begin{cases} > 0 & \text{for } 0 < u < \gamma, \\ = 0 & \text{for } u = \gamma, \\ < 0 & \text{for } u > \gamma, \end{cases} \\ \lim_{u \rightarrow \infty} G_a(u) &= \lim_{u \rightarrow \infty} \left[ \int_0^\gamma t^3 f''_a(t) dt + \int_\gamma^u t^3 f''_a(t) dt \right] < \gamma^3 \lim_{u \rightarrow \infty} \int_0^u f''_a(t) dt \\ &= \gamma^3 \left[ \lim_{u \rightarrow \infty} f'_a(u) - f'_a(0) \right] = \gamma^3 \left[ \lim_{u \rightarrow \infty} \frac{a^2 f_a(u)}{(a+u)^2} - f'_a(0) \right] \\ &= -\gamma^3 f'_a(0) < 0. \end{aligned}$$

So for any  $a > 4$ , there exists a unique number  $\kappa(a) > \gamma$  such that (3.15) holds. Since  $G'_a(\kappa(a)) = [\kappa(a)]^3 f''_a(\kappa(a)) < 0$  by (3.3), and by the Implicit Function Theorem,  $\kappa(a)$  is a continuous function of  $a$  on  $(4, \infty)$ . In addition, it is easy to observe that  $\gamma'(a) = a - 1 > 0$  for  $a > 4$ , and

$$\frac{\partial}{\partial t} \frac{t^2 f_a(t)}{(a+t)^3} = \frac{[-t^2 + (a^2 + a)t + 2a^2] t f_a(t)}{(a+t)^5} > 0 \quad \text{for } 0 \leq t \leq 8 \text{ and } a > 4.$$

Thus  $0 < \gamma(a) < \gamma(5) = 7.5 < 8$  for  $4 < a \leq \frac{417}{100}$ . Then we compute that

$$\begin{aligned} G_a(8) &= - \int_0^8 t^2 \theta''_a(t) dt = \int_0^8 t^3 f''_a(t) dt = 2a^2 \int_0^8 \left[ \frac{t^2 f_a(t)}{(a+t)^3} \left( \frac{\gamma t - t^2}{a+t} \right) \right] dt \\ &\leq \frac{2a^2 \gamma^2 f_a(\gamma)}{(a+\gamma)^3} \left[ \int_0^\gamma \left( \frac{\gamma t - t^2}{a+t} \right) dt + \int_\gamma^8 \left( \frac{\gamma t - t^2}{a+t} \right) dt \right] \\ &= \frac{2a^2 \gamma^2 f_a(\gamma)}{(a+\gamma)^3} \int_0^8 \left( \frac{\gamma t - t^2}{a+t} \right) dt = \frac{a^5 \gamma^2 f_a(\gamma)}{(a+\gamma)^3} \left[ \frac{8a^2 - 64}{a^3} + \ln \left( \frac{a}{a+8} \right) \right] \\ &= \frac{a^5 \gamma^2 f_a(\gamma)}{(a+\gamma)^3} \Psi_a, \end{aligned} \quad (3.21)$$

where  $\Psi_a \equiv (8a^2 - 64)/a^3 + \ln(a/(a+8))$ . Since

$$\frac{d}{da} \Psi_a = \frac{64(-a^2 + 3a + 24)}{a^4(a+8)} > 0 \quad \text{for } 4 < a \leq \frac{417}{100},$$

we see that  $\Psi_a \leq \Psi_{\frac{417}{100}} (\approx -0.03) < 0$  for  $4 < a \leq \frac{417}{100}$ . So by (3.21),  $G_a(8) < 0$  for  $4 < a \leq \frac{417}{100}$ . It implies that  $\kappa(a) < 8$  for  $4 < a \leq \frac{417}{100}$ .

In addition, by (3.3), we compute and find that, for  $4 < a \leq \frac{417}{100}$  and  $0 \leq u \leq 8$ ,

$$\begin{aligned} \frac{\partial}{\partial a} u^3 f_a''(u) &= \frac{u^3 a f_a(u)}{(a+u)^6} [(-4-2a)u^3 + (a^3+2a^2-6a)u^2 + 4a^3u + 2a^3] \\ &\geq \frac{u^3 a f_a(u)}{(a+u)^6} [(-4-2 \times \frac{417}{100})u^3 + (4^3+2 \times 4^2-6 \times \frac{417}{100})u^2 + 4 \times 4^3u + 2 \times 4^3] \\ &= \frac{u^3 a f_a(u)}{50(a+u)^6} [2u(309u+800)(8-u) + u^3 + 205u^2 + 6400] \\ &> 0. \end{aligned}$$

So for  $4 < a_1 < a_2 \leq \frac{417}{100}$ ,  $G_{a_1}(\kappa(a_2)) = \int_0^{\kappa(a_2)} t^3 f_{a_1}''(t) dt < \int_0^{\kappa(a_2)} t^3 f_{a_2}''(t) dt = G_{a_2}(\kappa(a_2)) = 0$ . It follows that  $\kappa(a_1) < \kappa(a_2)$  for  $4 < a_1 < a_2 \leq \frac{417}{100}$ . Therefore,  $\kappa(a)$  is a strictly increasing function of  $a \in (4, \frac{417}{100}]$ .

**Step 2.** We prove assertion (ii). It is easy to observe that  $t^2 - a^2t - a^2 < 0$  for  $0 \leq t \leq p_2(a) < a(a + \sqrt{a^2 + 4})/2$  and  $a > 4$ . So we further observe that

$$\frac{\partial}{\partial a} H_a(u) = \int_0^u (t^2 - a^2t - a^2) \frac{t^3 f_a(t)}{(a+t)^4} dt < 0 \quad \text{for } 0 < u \leq p_2(a) \text{ and } a > 4. \quad (3.22)$$

Clearly,  $H_a(0) = 0$ . By Lemma 3.4(i), we see that, for  $a > 4$ ,

$$H'_a(u) = u\theta'_a(u) \begin{cases} > 0 & \text{for } u \in (0, p_1(a)) \cup (p_2(a), \infty), \\ = 0 & \text{for } u \in \{p_1(a), p_2(a)\}, \\ < 0 & \text{for } u \in (p_1(a), p_2(a)). \end{cases} \quad (3.23)$$

So by (3.22) and (3.23), we find that

$$\frac{\partial}{\partial a} H_a(p_2(a)) = \frac{\partial}{\partial a} H_a(u) \Big|_{u=p_2(a)} + H'_a(p_2(a))p'_2(a) = \frac{\partial}{\partial a} H_a(u) \Big|_{u=p_2(a)} < 0. \quad (3.24)$$

Since  $H_a(p_2(a)) = 0$  for  $a = \tilde{a}$  by (1.4), and by (3.24), we observe that  $H(p_2(a)) < 0$  for  $a > \tilde{a}$ . So by (3.23), there exists a unique number  $\rho(a) \in (p_1(a), p_2(a)]$  such that (3.16) holds. Furthermore,  $\rho(a) = p_2(a)$  for  $a = \tilde{a}$  and  $p_1(a) < \rho(a) < p_2(a)$  for  $a > \tilde{a}$ . In addition, by integration by parts, we have

$$H_a(u) = \frac{1}{2} [u^2 \theta'_a(u) + G_a(u)]. \quad (3.25)$$

By (3.23) and (3.25),

$$G_a(\rho(a)) = -[\rho(a)]^2 \theta'_a(\rho(a)) \begin{cases} = 0 & \text{for } a = \tilde{a}, \\ > 0 & \text{for } a > \tilde{a}, \end{cases} \quad G_a(p_2(a)) = 2H_a(p_2(a)) \begin{cases} = 0 & \text{for } a = \tilde{a}, \\ < 0 & \text{for } a > \tilde{a}. \end{cases}$$

By (3.15), we see that  $\rho(a) = p_2(a) = \kappa(a)$  for  $a = \tilde{a}$  and  $\rho(a) < \kappa(a) < p_2(a)$  for  $a > \tilde{a}$ . Since  $\gamma'(a) > 0$  for  $a > 4$ , and by (3.6), (3.22) and (3.23), we find that

$$\frac{\partial}{\partial a} H_a(\gamma) = \frac{\partial}{\partial a} H_a(u) \Big|_{u=\gamma} + H'_a(\gamma)\gamma'(a) < \frac{\partial}{\partial a} H_a(u) \Big|_{u=\gamma} < 0 \quad \text{for } a > 4. \quad (3.26)$$

So we observe that  $H_a(\gamma)$  is a strictly decreasing continuous function of  $a > 4$ . By (1.3), we see that  $a^* = \inf \{a > 4 : H_a(\gamma) < 0\}$ . Thus  $H_a(\gamma) = 0$  for  $a = a^*$  and  $H_a(\gamma) < 0$  for  $a > a^*$ . It follows that  $\rho(a) = \gamma$  for  $a = a^*$ , and  $\rho(a) < \gamma$  for  $a > a^*$ . Thus, by above discussion, (3.17)–(3.20) hold. Next, we prove that  $\rho(a)$  is a strictly decreasing function of  $a \geq \tilde{a}$ . Let numbers  $a_2 > a_1 \geq \tilde{a}$  be given. Then  $\rho(a_1) \leq p_2(a_1) < p_2(a_2)$  by (3.9). So by (3.22), we observe that  $H_{a_2}(\rho(a_1)) < H_{a_1}(\rho(a_1)) = 0$ . It follows that  $\rho(a_2) < \rho(a_1)$ . It implies that  $\rho(a)$  is strictly decreasing for  $a \geq \tilde{a}$ .

Finally, the proof of the fact that  $\rho(a)$  is a continuous function of  $a \geq \tilde{a}$  is omitted.

The proof of Lemma 3.5 is complete. ■

### 3.2. Estimates of $T_a$ and its derivatives

For  $T_a(\alpha)$  in (3.1), we compute that

$$T'_a(\alpha) = \frac{1}{2\sqrt{2}\alpha} \int_0^\alpha \frac{\theta_a(\alpha) - \theta_a(u)}{[F_a(\alpha) - F_a(u)]^{3/2}} du \quad \text{for } 0 < \alpha < \infty, \quad (3.27)$$

where  $\theta_a(u)$  is defined by (3.4). The next Lemma 3.6(i) follows easily from [17, Lemma 3.2] and (3.3). And the proof of next Lemma 3.6(ii) is easy but tedious and hence we omit it.

**Lemma 3.6.** Consider (1.1) with fixed  $a > 0$ . The following assertions (i)–(ii) hold:

- (i) For any fixed  $a > 4$ , either  $T_a(\alpha)$  is strictly increasing on  $(0, \gamma]$ , or  $T_a(\alpha)$  is strictly increasing and then strictly decreasing on  $(0, \gamma]$ .
- (ii) For any fixed  $\alpha > 0$ ,  $T'_a(\alpha)$  is a continuously differentiable function of  $a > 0$ .

**Lemma 3.7.** Consider (1.1) with  $a > 4$ . Then

$$T''_a(\alpha) + \frac{2}{\alpha} T'_a(\alpha) > 0 \quad \text{for } \alpha \geq \kappa(a).$$

**Proof of Lemma 3.7.** By (3.27), we compute that

$$\begin{aligned} T''_a(\alpha) + \frac{2}{\alpha} T'_a(\alpha) &= \frac{1}{2\sqrt{2}\alpha^2} \int_0^\alpha \frac{\frac{3}{2} [\theta_a(\alpha) - \theta_a(u)]^2 + [F_a(\alpha) - F_a(u)][\phi_a(\alpha) - \phi_a(u)]}{[F_a(\alpha) - F_a(u)]^{5/2}} du \\ &\geq \frac{1}{2\sqrt{2}\alpha^2} \int_0^\alpha \frac{\phi_a(\alpha) - \phi_a(u)}{[F_a(\alpha) - F_a(u)]^{3/2}} du, \end{aligned} \quad (3.28)$$

where  $\phi_a(u) \equiv u\theta'_a(u) - \theta_a(u)$ , see [12, (3.12)]. We obtain that, by (3.3),

$$\phi_a(0) = 0 \quad \text{and} \quad \phi'_a(u) = u\theta''_a(u) = -u^2 f''_a(u) \quad \begin{cases} < 0 & \text{for } 0 < u < \gamma, \\ = 0 & \text{for } u = \gamma, \\ > 0 & \text{for } u > \gamma. \end{cases} \quad (3.29)$$

We note that  $\kappa(a) > \gamma$  for  $a > 4$  by Lemma 3.5(i). We fix  $\alpha \geq \kappa(a)$ . If  $\phi_a(\alpha) \geq 0$ , by (3.29), we see that  $\phi_a(\alpha) - \phi_a(u) > 0$  for  $0 < u < \alpha$ , and hence  $T''_a(\alpha) + \frac{2}{\alpha} T'_a(\alpha) > 0$  by (3.28). While if  $\phi_a(\alpha) < 0$ , since  $\alpha \geq \kappa(a) > \gamma$ , there exists  $\xi_\alpha \in (0, \gamma)$  such that  $\phi_a(\xi_\alpha) = \phi_a(\alpha)$ . See Figure 3.3. So by [12, (3.15)] and Lemma 3.5(i),

$$T''_a(\alpha) + \frac{2}{\alpha} T'_a(\alpha) > \frac{-1}{2\sqrt{2}\alpha^2 [F_a(\alpha) - F_a(\xi_\alpha)]^{3/2}} \int_0^\alpha u^3 f''_a(u) du \geq 0.$$

The proof of Lemma 3.7 is complete. ■

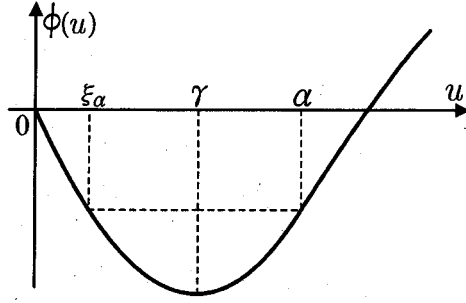


Figure 3.3: The graph of  $\phi_a(u)$  with  $\phi_a(\alpha) < 0$  and  $\alpha > \gamma > \xi_\alpha > 0$ .

**Lemma 3.8.** Consider (1.1) with  $a \geq \bar{a}$ . Then  $T'_a(\alpha) < 0$  for  $\rho(a) \leq \alpha \leq p_2(a)$ . In particular,  $\rho(a) < 3$  for  $a \geq 6$ .

**Proof of Lemma 3.8.** Let  $\alpha \in [\rho(a), p_2(a)]$  be given. By Lemmas 3.4 and 3.5(ii), we observe that

$$0 < \bar{\alpha} < p_1(a) < \rho(a) \leq \alpha \leq p_2(a) \text{ for } a \geq \bar{a}.$$

Moreover,  $\theta_a(\alpha) - \theta_a(u) > 0$  for  $0 < u < \bar{\alpha}$  and  $\theta_a(\alpha) - \theta_a(u) < 0$  for  $\bar{\alpha} < u < \alpha$ . Then by Lemma 3.5(ii), we obtain that

$$\begin{aligned} T'_a(\alpha) &= \frac{1}{2\sqrt{2}\alpha} \left\{ \int_0^{\bar{\alpha}} \frac{\theta_a(\alpha) - \theta_a(u)}{[F_a(\alpha) - F_a(u)]^{3/2}} du + \int_{\bar{\alpha}}^\alpha \frac{\theta_a(\alpha) - \theta_a(u)}{[F_a(\alpha) - F_a(u)]^{3/2}} du \right\} \\ &< \frac{1}{2\sqrt{2}\alpha} \left\{ \int_0^{\bar{\alpha}} \frac{\theta_a(\alpha) - \theta_a(u)}{[F_a(\alpha) - F_a(\bar{\alpha})]^{3/2}} du + \int_{\bar{\alpha}}^\alpha \frac{\theta_a(\alpha) - \theta_a(u)}{[F_a(\alpha) - F_a(\bar{\alpha})]^{3/2}} du \right\} \\ &= \frac{1}{2\sqrt{2}\alpha [F_a(\alpha) - F_a(\bar{\alpha})]^{3/2}} \int_0^\alpha [\theta_a(\alpha) - \theta_a(u)] du \\ &= \frac{1}{2\sqrt{2}\alpha [F_a(\alpha) - F_a(\bar{\alpha})]^{3/2}} \left[ \alpha \theta_a(\alpha) - \int_0^\alpha \theta_a(u) du \right] \\ &= \frac{1}{2\sqrt{2}\alpha [F_a(\alpha) - F_a(\bar{\alpha})]^{3/2}} \int_0^\alpha u \theta'_a(u) du \\ &= \frac{1}{2\sqrt{2}\alpha [F_a(\alpha) - F_a(\bar{\alpha})]^{3/2}} \int_0^\alpha [u f_a(u) - u^2 f'_a(u)] du < 0. \end{aligned}$$

Next, we prove that  $\rho(a) < 3$  for  $a \geq 6$ . It is easy to compute and find that

$$p'_1(a) = \frac{(a-1)\sqrt{a^2-4a} - a(a-3)}{\sqrt{a^2-4a}} < 0 \text{ for } a > 4. \quad (3.30)$$

So by (3.9), we compute and find that

$$p_2(a) \geq p_2(6) (\approx 22.3) > 3 > p_1(6) (\approx 1.6) > p_1(a) \text{ for } a \geq 6. \quad (3.31)$$

By Lemma 3.4(i) and (3.31),

$$u\theta'_a(u) \begin{cases} > 0 & \text{for } 0 < u < p_1(a), \\ = 0 & \text{for } u = p_1(a), \\ < 0 & \text{for } p_1(a) < u < 3. \end{cases} \quad (3.32)$$

We compute that

$$\int \frac{u(a+u)^2 - a^2u^2}{(a+u)^2} du = \frac{u^2}{2} - a^2u + \frac{a^4}{a+u} + 2a^3 \ln(a+u). \quad (3.33)$$

By (3.2) and (3.31)–(3.33), we compute and observe that, for  $a \geq 6$ ,

$$\begin{aligned} H_a(3) &= \int_0^3 u\theta'_a(u) du = \int_0^{p_1(a)} u\theta'_a(u) du + \int_{p_1(a)}^3 u\theta'_a(u) du \\ &\leq f_a(p_1(a)) \left[ \int_0^{p_1(a)} \frac{u(a+u)^2 - a^2u^2}{(a+u)^2} du + \int_{p_1(a)}^3 \frac{u(a+u)^2 - a^2u^2}{(a+u)^2} du \right] \\ &= f_a(p_1(a)) \int_0^3 \frac{u(a+u)^2 - a^2u^2}{(a+u)^2} du = \frac{f_a(p_1(a))}{8a^3(a+3)^2} \Lambda_a, \end{aligned} \quad (3.34)$$

where

$$\Lambda_a \equiv \frac{-12a^3 - 18a^2 + 9a + 27}{4a^3(a+3)} - \ln\left(\frac{a}{a+3}\right).$$

We find that  $\Lambda_6 = -\frac{13}{32} - \ln \frac{2}{3} (\approx -7.85 \times 10^{-4}) < 0$ ,  $\lim_{a \rightarrow \infty} \Lambda_a = 0$ , and

$$\Lambda'(a) = \frac{27}{4a^4(a+3)^2} (a^2 - 6a - 9) \begin{cases} < 0 & \text{for } 6 \leq a < 3 + 3\sqrt{2}, \\ = 0 & \text{for } a = 3 + 3\sqrt{2}, \\ > 0 & \text{for } a > 3 + 3\sqrt{2}. \end{cases}$$

So by (3.34),

$$H_a(3) \leq \frac{f_a(p_1(a))}{8a^3(a+3)^2} \Lambda_a < 0 \quad \text{for } a \geq 6,$$

which implies that  $\rho(a) < 3$  for  $a \geq 6$ . The proof of Lemma 3.8 is complete. ■

**Lemma 3.9.** Consider (1.1) with  $a > 0$ . Then

$$\alpha f_a(\alpha) - u f_a(u) \leq M_a(u, \alpha) [F_a(\alpha) - F_a(u)] \quad \text{for } 0 \leq u \leq \alpha,$$

where

$$M_a(u, \alpha) \equiv \begin{cases} 1 + \frac{a^2\alpha}{(a+\alpha)^2} & \text{for } 0 \leq u \leq \alpha \leq a, \\ 1 + \frac{a}{4} & \text{for } 0 \leq u \leq a < \alpha, \\ 1 + \frac{a^2u}{(a+u)^2} & \text{otherwise} \end{cases}$$

satisfies  $M_a(u, \alpha) \leq 1 + \frac{a}{4}$  for  $u \geq 0$  and  $\alpha \geq 0$ .

**Proof of Lemma 3.9** Since we compute that, for any  $a > 0$ ,

$$\frac{\partial}{\partial u} \frac{a^2u}{(a+u)^2} = \frac{a^2(a-u)}{(a+u)^3} \begin{cases} > 0 & \text{if } 0 < u < a, \\ = 0 & \text{if } u = a, \\ < 0 & \text{if } a > u, \end{cases} \quad (3.35)$$

we see that

$$M_a(u, \alpha) \leq 1 + \frac{a^2 \times a}{(a + a)^2} = 1 + \frac{a}{4} \text{ for } u \geq 0 \text{ and } \alpha \geq 0.$$

We let

$$W_a(u, \alpha) \equiv M_a(u, \alpha) [F_a(\alpha) - F_a(u)] - [\alpha f_a(\alpha) - u f_a(u)] \text{ for } u \geq 0.$$

To complete the proof, it is sufficient to prove that  $W_1(u, \alpha) \geq 0$  for  $0 \leq u \leq \alpha$  and  $a > 0$ . We compute and obtain that

$$\begin{aligned} & \frac{\partial}{\partial u} W_a(u, \alpha) \\ &= \left[ \frac{\partial}{\partial u} M_a(u, \alpha) \right] [F_a(\alpha) - F_a(u)] + \left[ 1 + \frac{a^2 u}{(a + u)^2} - M_a(u, \alpha) \right] f_a(u) \\ &= \begin{cases} \left[ 1 + \frac{a^2 u}{(a + u)^2} - M_a(u, \alpha) \right] f_a(u) = \left[ \frac{a^2 u}{(a + u)^2} - \frac{a^2 \alpha}{(a + \alpha)^2} \right] f_a(u) < 0 & \text{if } 0 \leq u < \alpha \leq a, \\ \left[ 1 + \frac{a^2 u}{(a + u)^2} - M_a(u, \alpha) \right] f_a(u) = \left[ \frac{a^2 u}{(a + u)^2} - \frac{a}{4} \right] f_a(u) < 0 & \text{if } 0 \leq u \leq a < \alpha, \\ \left[ \frac{\partial}{\partial u} M_a(u, \alpha) \right] [F_a(\alpha) - F_a(u)] = \frac{a^2(a - u)}{(a + u)^4} [F_a(\alpha) - F_a(u)] < 0 & \text{if } 0 < a < u < \alpha. \end{cases} \quad (3.36) \end{aligned}$$

Since  $W_a(\alpha, \alpha) = 0$  and by (3.36), we see that  $W_a(u, \alpha) \geq 0$  for  $0 \leq u \leq \alpha$ . The proof of Lemma 3.9 is complete. ■

The proof of the following Lemma 3.10 is rather lengthy, and hence it is given in [11].

**Lemma 3.10.** Consider (1.1) with  $4 < a < \tilde{a}$ . Then  $[\alpha T_a''(\alpha)]' > 0$  for  $\gamma(a) \leq \alpha \leq \kappa(a) = \eta(a)$ .

**Lemma 3.11.** Consider (1.1) with  $4 < a < a^*$ . Then  $[\alpha T_a'''(\alpha)]' > 0$  for

$$\gamma(a) \leq \alpha \leq \eta(a) \equiv \begin{cases} \kappa(a) & \text{for } 4 < a < \tilde{a}, \\ \rho(a) & \text{for } a \geq \tilde{a}. \end{cases} \quad (3.37)$$

Moreover, one of the following assertions (a)–(c) holds:

- (a)  $T_a'(\alpha)$  is a strictly increasing function of  $\alpha$  on  $[\gamma(a), \eta(a)]$ .
- (b)  $T_a'(\alpha)$  is a strictly decreasing function of  $\alpha$  on  $[\gamma(a), \eta(a)]$ .
- (c)  $T_a'(\alpha)$  is a strictly decreasing and then strictly increasing function of  $\alpha$  on  $[\gamma(a), \eta(a)]$ .

**Proof of Lemma 3.11.** By [10, Lemma 2.6], we obtain that  $[\alpha T_a''(\alpha)]' > 0$  for  $\gamma(a) \leq \alpha \leq \rho(a) = \eta(a)$  and  $4 < \tilde{a} \leq a < a^*$ . By Lemma 3.10, we see that  $[\alpha T_a''(\alpha)]' > 0$  for  $\gamma(a) \leq \alpha \leq \eta(a)$  and  $4 < a < \tilde{a}$ . So  $[\alpha T_a''(\alpha)]' > 0$  for  $\gamma(a) \leq \alpha \leq \eta(a)$  and  $4 < a < a^*$ . By Lemma 3.5, we see that  $\gamma(a) < \eta(a)$  for  $4 < a < a^*$ . Since  $\alpha T_a''(\alpha)$  is a strictly increasing function of  $\alpha \in [\gamma(a), \eta(a)]$  for  $4 < a < a^*$ , we observe that there are three possible cases:

Case 1  $T_a''(\alpha) > 0$  for  $\alpha \in (\gamma(a), \eta(a))$ .

Case 2  $T_a''(\alpha) < 0$  for  $\alpha \in [\gamma(a), \eta(a)]$ .

Case 3  $T_a''(\alpha) < 0$  for  $\alpha \in [\gamma(a), \alpha']$ ,  $T_a''(\alpha) > 0$  for  $\alpha \in (\alpha', \eta(a))$ , and  $T_a''(\alpha') = 0$  for some  $\alpha' \in (\gamma(a), \eta(a))$ .

So if Case 1 (Case 2 and Case 3 respectively) holds, then assertion (a) ((b) and (c) respectively) holds.

The proof of Lemma 3.11 is complete. ■

**Remark 1.** By (3.37) and Lemma 3.5, we see that  $\eta(a)$  is a continuous function of  $a > 4$ .

**Lemma 3.12.** Consider (1.1) with  $a > 4$ . The following assertions (i)–(ii) hold:

(i)

$$\eta(a) < \omega(a) \equiv \begin{cases} 12 & \text{if } 4 < a < 6, \\ 3 & \text{if } a \geq 6. \end{cases}$$

(ii)  $\partial T'_a(\alpha)/\partial a < 0$  for  $0 < \alpha \leq \omega(a)$ .

**Proof of Lemma 3.12.** By (3.9), Lemmas 3.5 and 3.8, we see that

$$\eta(a) = \begin{cases} \kappa(a) < 8 < 12 = \omega(a) & \text{for } 4 < a < \tilde{a}, \\ \rho(a) \leq p_2(a) < p_2(a^*) < 12 = \omega(a) & \text{for } \tilde{a} \leq a < a^*, \\ \rho(a) \leq \gamma(a) < \gamma(6) = 12 = \omega(a) & \text{for } a^* \leq a < 6, \\ \rho(a) < 3 = \omega(a) & \text{for } a \geq 6. \end{cases} \quad (3.38)$$

So  $\eta(a) < \omega(a)$ , and hence assertion (i) holds.

We compute that

$$\frac{\partial}{\partial a} T'_a(\alpha) = \frac{1}{2\sqrt{2}\alpha} \int_0^\alpha \frac{N_a(u, \alpha)}{[F_a(\alpha) - F_a(u)]^{3/2}} du,$$

where

$$\begin{aligned} N_a(u, \alpha) &\equiv [F_a(\alpha) - F_a(u)] \left[ \frac{u^3 f_a(u)}{(a+u)^2} - \frac{\alpha^3 f_a(\alpha)}{(a+\alpha)^2} - \int_u^\alpha \frac{t^2 f_a(t)}{(a+t)^2} dt \right] \\ &\quad + \frac{3}{2} [\alpha f_a(\alpha) - u f_a(u)] \int_u^\alpha \frac{t^2 f_a(t)}{(a+t)^2} dt. \end{aligned} \quad (3.39)$$

By (3.12), (3.39) and Lemma 3.9, we obtain that

$$\begin{aligned} \frac{N_a(u, \alpha)}{[\alpha f_a(\alpha) - u f_a(u)]} &\leq \frac{3}{2} \int_u^\alpha \frac{t^2 f_a(t)}{(a+t)^2} dt - \frac{1}{M_a(u, \alpha)} \int_u^\alpha \frac{[2t^2 + a(a+6)t + 4a^2] t^2 f_a(t)}{(a+t)^4} dt \\ &\equiv N_{a,1}(u, \alpha). \end{aligned} \quad (3.40)$$

Thus, to prove assertion (ii), it is sufficient to prove the next parts (a) and (b):

(a) For  $4 < a < 6$ ,  $N_{a,1}(u, \alpha) < 0$  for  $0 < u < \alpha \leq 12$ .

(b) For  $a \geq 6$ ,  $N_{a,1}(u, \alpha) < 0$  for  $0 < u < \alpha \leq 3$ .

(I) We prove part (a). Assume that  $4 < a < 6$ . By Lemma 3.9, we have that  $M_a(u, \alpha) \leq (a+4)/4$  for  $0 \leq u \leq \alpha$ . We compute and find that, for  $4 < a < 6$ ,

$$(3a-4)12^2 - 2a(a-12)12 + a^2(3a-20) = -(12-a)[3a^2 + 8(6-a)] < 0.$$



It follows that  $(3a - 4)t^2 - 2a(a - 12)t + a^2(3a - 20) < 0$  for  $0 \leq t \leq 12$  and  $4 < a < 6$ . So by (3.40), we obtain that, for  $0 \leq u < \alpha \leq 12$  and  $4 < a < 6$ ,

$$\begin{aligned} N_{a,1}(u, \alpha) &\leq \frac{3}{2} \int_u^\alpha \frac{t^2 f_a(t)}{(a+t)^2} dt - \frac{4}{a+4} \int_u^\alpha \frac{[2t^2 + a(a+6)t + 4a^2] t^2 f_a(t)}{(a+t)^4} dt \\ &= \int_u^\alpha \frac{t^2 f_a(t)}{2(a+4)(a+t)^4} [(3a-4)t^2 - 2a(a-12)t + a^2(3a-20)] dt < 0. \end{aligned}$$

So part (a) holds.

(II) We prove part (b). Assume that  $a \geq 6$ . Since  $a > 3 \geq \alpha > 0$  and by Lemma 3.9, we observe that

$$\frac{1}{M_a(u, \alpha)} = \frac{(a+\alpha)^2}{(a+\alpha)^2 + a^2\alpha} \quad \text{for } 0 \leq u \leq \alpha.$$

So by (3.40), we obtain that

$$N_{a,1}(u, \alpha) = \int_u^\alpha \frac{t^2 f_a(t)}{(a+t)^2} N_{a,2}(t, \alpha) dt, \quad (3.41)$$

where

$$N_{a,2}(t, \alpha) \equiv \frac{3}{2} - \frac{(a+\alpha)^2}{(a+\alpha)^2 + a^2\alpha} \left[ \frac{2t^2 + a(a+6)t + 4a^2}{(a+t)^2} \right].$$

We compute and observe that, for  $a > 6$  and  $0 \leq t \leq 3$ ,

$$\begin{aligned} \frac{\partial}{\partial t} N_{a,2}(t, \alpha) &= \frac{a(a+\alpha)^2}{[(a+\alpha)^2 + a^2\alpha] (a+t)^3} [(a+2)t - a^2 + 2a] \\ &\leq \frac{a(a+\alpha)^2}{[(a+\alpha)^2 + a^2\alpha] (a+t)^3} [(a+2) \times 3 - a^2 + 2a] \\ &= \frac{a(a+\alpha)^2(a+1)(6-a)}{[(a+\alpha)^2 + a^2\alpha] (a+t)^3} < 0. \end{aligned}$$

So  $N_{a,2}(t, \alpha)$  is a strictly decreasing function of  $t \in [0, 3]$  for  $a \geq 6$ . Since

$$N_{a,2}(\alpha, \alpha) = \frac{(\alpha-5)a^2 - 6a\alpha - \alpha^2}{2(a+\alpha)^2 + 2a^2\alpha} < 0 \quad \text{for } 0 < \alpha \leq 3,$$

we see that either  $N_{a,2}(t, \alpha) < 0$  for  $0 < t \leq \alpha$ , or

$$N_{a,2}(t, \alpha) = \begin{cases} > 0 & \text{if } 0 < t < r_1, \\ = 0 & \text{if } t = r_1, \\ < 0 & \text{if } r_2 < t \leq \alpha \end{cases} \quad \text{for some } r_1 \in (0, \alpha).$$

So by (3.41), we further see that, for  $0 < u < \alpha \leq 3$ ,

$$\text{either } \frac{\partial}{\partial u} N_{a,1}(u, \alpha) > 0 \quad \text{or} \quad \frac{\partial}{\partial u} N_{a,1}(u, \alpha) \begin{cases} < 0 & \text{if } 0 < t < r_1, \\ = 0 & \text{if } t = r_1, \\ > 0 & \text{if } r_2 < t \leq \alpha. \end{cases} \quad (3.42)$$

In addition, since  $3\alpha^2 - 8\alpha - 5 < 0$  for  $0 < \alpha \leq 3$ , and by (3.41), we compute and find that, for  $a \geq 6$ ,

$$\begin{aligned} \frac{\partial}{\partial \alpha} \left[ \frac{2(a+\alpha)^2 + 2a^2\alpha}{(a+\alpha)^2 + 3a^2\alpha} N_{a,1}(0, \alpha) \right] &= \frac{\alpha^2 f_a(\alpha)}{\left[ (a+\alpha)^2 + 3a^2\alpha \right]^2 (a+\alpha)^2} [(3\alpha^2 - 8\alpha - 5)a^4 \\ &\quad - 16(\alpha^2 + \alpha)a^3 - (8\alpha + 18)\alpha^2 a^2 - 8\alpha^3 a - \alpha^4] \\ &< 0. \end{aligned}$$

It follows that

$$\frac{2(a+\alpha)^2 + 2a^2\alpha}{(a+\alpha)^2 + 3a^2\alpha} N_{a,1}(0, \alpha) < \frac{2(a+\alpha)^2 + 2a^2\alpha}{(a+\alpha)^2 + 3a^2\alpha} N_{a,1}(0, \alpha) \Big|_{\alpha=0} = 0 \quad \text{for } 0 < \alpha \leq 3.$$

Thus,  $N_{a,1}(0, \alpha) < 0$  for  $0 < \alpha \leq 3$ . Clearly,  $N_{a,1}(\alpha, \alpha) = 0$ . So by (3.42),  $N_{a,1}(u, \alpha) < 0$  for  $0 \leq u < \alpha$  and  $0 < \alpha \leq 3$ . So part (b) holds.

The proof of Lemma 3.12 is complete. ■

### 3.3. Statements and proofs of main lemmas

**Lemma 3.13.** Consider (1.1) with fixed  $a > 4$ . Either one of the following assertions (i)–(ii) holds:

- (i)  $T_a(\alpha)$  is a strictly increasing function on  $(0, \infty)$ .
- (ii)  $T_a(\alpha)$  has exactly one local maximum and exactly one local minimum on  $(0, \infty)$ .

**Proof of Lemma 3.13.** Assume that assertion (i) does not hold. Then by Lemma 3.1,  $T_a(\alpha)$  has a local maximum and a local minimum on  $(0, \infty)$ .

Assume that  $T_a(\alpha)$  has two local maxima at some  $\alpha_{M_1} < \alpha_{M_2}$ . Then there exists  $\alpha_m \in (\alpha_{M_1}, \alpha_{M_2})$  such that  $T_a(\alpha_m)$  is a local minimum value. If  $4 < a < \tilde{a}$ , and by Lemmas 3.5(ii) and 3.6–3.8, we observe that  $\gamma(a) \leq \alpha_m < \alpha_{M_2} < \kappa(a) = \eta(a)$ . It is a contradiction by Lemma 3.11. If  $\tilde{a} \leq a < a^*$ , and by Lemmas 3.5(ii) and 3.6–3.8, we observe that  $\gamma(a) \leq \alpha_m < \alpha_{M_2} < \rho(a) = \eta(a)$ . It is a contradiction by Lemma 3.11. If  $a \geq a^*$ , and by Lemmas 3.5(ii), 3.7 and 3.8, we observe that  $\alpha_m < \alpha_{M_2} \leq \gamma(a)$ . It is a contradiction by Lemma 3.6(i). So by above discussions,  $T_a(\alpha)$  has exactly one local maximum on  $(0, \infty)$ .

In addition, if  $T_a(\alpha)$  has two local minima at some  $\alpha_{m_1} < \alpha_{m_2}$ , then by Lemma 3.1,  $T_a(\alpha)$  has two local maxima at some  $\alpha_{M_1} \in (0, \alpha_{m_1})$  and  $\alpha_{M_2} \in (\alpha_{m_1}, \alpha_{m_2})$ . It is a contradiction. So  $T_a(\alpha)$  has exactly one local minimum on  $(0, \infty)$ .

The proof of Lemma 3.13 is complete. ■

**Lemma 3.14.** Consider (1.1) with  $a > 4$ . Either one of the following two assertions holds:

- (i)  $T_a(\alpha)$  is a strictly increasing function on  $(0, \infty)$  and  $T_a(\alpha)$  has at most one critical point on  $(0, \infty)$ .
- (ii)  $T_a(\alpha)$  has exactly two critical points, a local maximum at some  $\alpha_M$  and a local minimum at some  $\alpha_m > \alpha_M$  on  $(0, \infty)$ .

**Proof of Lemma 3.14.** By Lemma 3.13, either one of the following two cases holds:

- (a)  $T_a(\alpha)$  is a strictly increasing function on  $(0, \infty)$ .

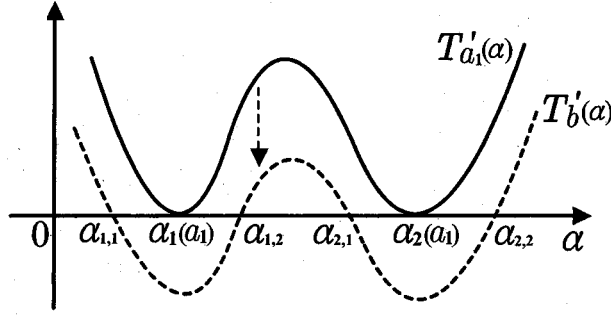


Figure 3.4: Graphs of  $T'_{a_1}(\alpha)$  and  $T'_b(\alpha)$  with  $b < a_1$  sufficiently close to  $a_1$ .

- (b)  $T_a(\alpha)$  has exactly one local maximum at some  $\alpha_M$  and exactly one local minimum at some  $\alpha_m (> \alpha_M)$  on  $(0, \infty)$ .

(I) We prove assertion (i) under Case (a). We fix  $a_1 > 4$ . Assume that  $T_{a_1}(\alpha)$  has two critical points  $\alpha_1(a_1) < \alpha_2(a_1)$  on  $(0, \infty)$ . We obtain that

$$T'_{a_1}(\alpha_1(a_1)) = T'_{a_1}(\alpha_2(a_1)) = T''_{a_1}(\alpha_1(a_1)) = T''_{a_1}(\alpha_2(a_1)) = 0. \quad (3.43)$$

So by Lemmas 3.7 and 3.8, we observe that  $0 < \alpha_1(a_1) < \alpha_2(a_1) < \eta(a_1) < \omega(a_1)$ . By (3.38) and continuities of  $\eta(a)$ ,  $\omega(a)$  and  $p_2(a)$ , we observe that

$$0 < \alpha_1(a_1) < \alpha_2(a_1) < \min_{a \in [a_1 - \delta, a_1]} \{\omega(a)\} \text{ for some } \delta > 0. \quad (3.44)$$

Let  $b \in (a_1 - \delta, a_1)$  be given. By Lemma 3.12, (3.43) and (3.44), we observe that

$$T'_b(\alpha_1(a_1)) < T'_{a_1}(\alpha_1(a_1)) = 0 \text{ and } T'_b(\alpha_2(a_1)) < T'_{a_1}(\alpha_2(a_1)) = 0. \quad (3.45)$$

In addition, we assume that there exists an open interval  $I$  such that  $T'_a(\alpha) \equiv 0$  on  $I$ . It implies that  $T''_a(\alpha) = T'''_a(\alpha) = 0$  on  $I$ . It is a contradiction by Lemmas 3.6–3.8 and 3.11. So there exist three numbers  $\beta_1 \in (0, \alpha_1(a_1))$ ,  $\beta_2 \in (\alpha_1(a_1), \alpha_2(a_1))$  and  $\beta_3 \in (\alpha_2(a_1), \eta(a_1))$  such that  $T'_{a_1}(\beta_i) > 0$  for  $i = 1, 2, 3$ . So by Lemma 3.6(ii) and (3.45), we choose  $b < a_1$  sufficiently close to  $a_1$  such that  $T'_b(\alpha)$  has four positive zeros  $\alpha_{1,1}, \alpha_{1,2}, \alpha_{2,1}, \alpha_{2,2}$  satisfying

$$\alpha_{1,1} < \alpha_1(a_1) < \alpha_{1,2} < \alpha_{2,1} < \alpha_2(a_1) < \alpha_{2,2}.$$

See Figure 3.4. Furthermore,  $T_b(\alpha_{1,1})$  and  $T_b(\alpha_{2,1})$  are local maximum values, and  $T_b(\alpha_{1,2})$  and  $T_b(\alpha_{2,2})$  are local minimum values. It is a contradiction by Lemma 3.13. Therefore, assertion (i) holds under Case (a).

(II) We prove assertion (ii) under Case (b). We fix  $a_2 > 4$ . Assume that  $T_{a_2}(\alpha)$  has a critical point  $\alpha_3(a_2)$  on  $(0, \infty)$ , distinct from  $\alpha_M$  and  $\alpha_m$ . It follows that  $T'_{a_2}(\alpha_3(a_2)) = T''_{a_2}(\alpha_3(a_2)) = 0$ . By Lemmas 3.7 and 3.8, we obtain that  $0 < \alpha_3(a_2) < \eta(a_2) < \omega(a_2)$ . Similarly, we have that

$$0 < \alpha_3(a_2) < \min_{a \in [a_2 - \delta, a_2 + \delta]} \{\omega(a)\} \text{ for some } \delta > 0.$$

So by Lemma 3.12, we observe that

$$T'_b(\alpha_3(a_2)) < T'_{a_2}(\alpha_3(a_2)) = 0 \quad \text{for } a_2 - \delta < b < a_2, \quad (3.46)$$

$$T'_b(\alpha_3(a_2)) > T'_{a_2}(\alpha_3(a_2)) = 0 \quad \text{for } a_2 < b < a_2 + \delta. \quad (3.47)$$

Similarly, by Lemma 3.6(ii), (3.46) and (3.47), there exists  $b > 0$  sufficiently close to  $a_2$  such that  $T_b(\alpha)$  has two local extrema at  $\alpha_{3,1} \in (0, \alpha_3(a_2))$  and  $\alpha_{3,2} \in (\alpha_3(a_2), \eta(a_2))$ , distinct from  $\alpha_M$  and  $\alpha_m$ . See Figure 3.5. It is a contradiction by Lemma 3.13. Therefore, assertion (ii) holds under

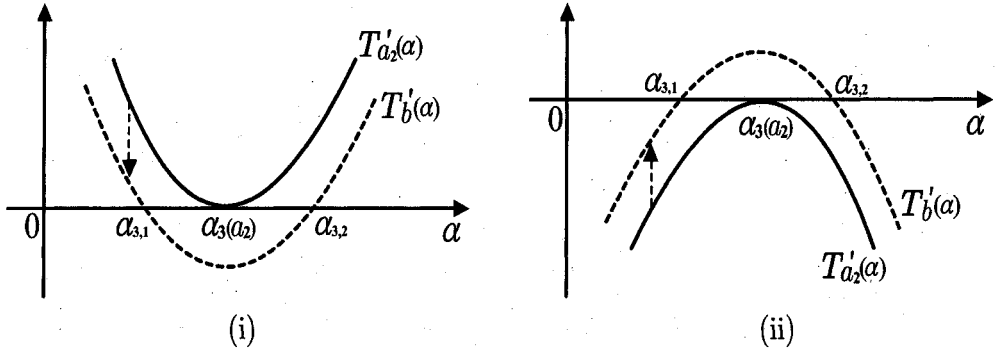


Figure 3.5: Local graphs of  $T'_{a_2}(\alpha)$  and  $T'_b(\alpha)$  for  $\alpha$  near  $\alpha_3(a_2)$  and  $b > 0$  sufficiently close to  $a_2$ . (i)  $T''_{a_2}(\alpha_3(a_2)) \geq 0$ . (ii)  $T''_{a_2}(\alpha_3(a_2)) \leq 0$ .

Case (b).

The proof of Lemma 3.14 is complete. ■

We are in a position to prove Lemma 3.2 by applying Lemmas 3.5(i), 3.6(ii), 3.7 and 3.12.

**Proof of Lemma 3.2.** By Theorem 1.2 and Lemma 3.14, we obtain that

$$\begin{aligned} \Omega &= \left\{ a > 0 : T_a(\alpha) \text{ has exactly two critical points,} \right. \\ &\quad \left. \text{a local maximum and a local minimum, on } (0, \infty) \right\} \\ &= \{ a > 4 : T'_a(\alpha) < 0 \text{ for some } \alpha \in (0, \infty) \}. \end{aligned} \quad (3.48)$$

(I) It is obvious that  $\Omega$  is nonempty because  $[\bar{a}, \infty) \subset \Omega$  by [12, Theorem 2.2] and Lemma 3.14.

(II) We show that  $\Omega$  is open. If  $b \in \Omega$ , then  $T'_b(\beta) < 0$  for some  $\beta \in (0, \infty)$ . By Lemma 3.6(ii), we observe that  $T'_a(\beta) < 0$  for  $a$  belonging to some open neighborhood of  $b$ . So  $\Omega$  is open.

(III) We then show that  $\Omega$  is connected. First, we see that  $[\bar{a}, \infty) \subset \Omega$  by [12, Theorem 2.2(i)]. Suppose to the contrary that the set  $\Omega \cap (4, \bar{a}]$  is not connected, then there exist positive numbers  $a_1$  and  $a_2$  satisfying  $4 < a_1 < a_2 < \bar{a}$  such that  $a_1 \in \Omega$  and  $a_2 \notin \Omega$ . Hence  $T'_{a_2}(\alpha) \geq 0$  on  $(0, \infty)$  by Lemma 3.14 and (3.48). Since  $\bar{a} (\approx 4.166) < \frac{417}{100}$  by (1.4), and by Lemma 3.5(i), we have that  $\kappa(a_1) < \kappa(a_2) < 8$ . So by Lemma 3.12,

$$T'_{a_1}(\alpha) > T'_{a_2}(\alpha) \geq 0 \quad \text{for all } \alpha \in (0, \kappa(a_1)) \subset (0, \kappa(a_2)). \quad (3.49)$$

Since  $a_1 \in \Omega \cap (4, \bar{a})$  and by (3.49), there exists a number  $\alpha_1 \geq \kappa(a_1)$  such that  $T'_{a_1}(\alpha_1) = 0$  and  $T''_{a_1}(\alpha_1) \leq 0$ . It is a contradiction by Lemma 3.7. So  $\Omega \cap (4, \bar{a}]$  is connected. It implies that  $\Omega$  is connect.

(IV) Since  $\Omega$  is open and connect and  $[\tilde{a}, \infty) \subset \Omega$ , there exists  $a_0 \in (4, \tilde{a})$  such that  $\Omega = (a_0, \infty)$ . Moreover, by numerical simulation, we find that  $a_0 \approx 4.069$ .

The proof of Lemma 3.2 is complete. ■

In addition to  $T_a(\alpha)$  with  $0 < a < \infty$  defined in (3.1) to problem (1.1) with  $f_a(u) = \exp\left(\frac{au}{a+u}\right)$ , we define two time-map functions  $T_0(\alpha)$  and  $T_\infty(\alpha)$  for corresponding nonlinearities  $f_0(u) \equiv 1$  and  $f_\infty(u) = \exp u$  by

$$T_0(\alpha) \equiv \frac{1}{\sqrt{2}} \int_0^\alpha \frac{1}{\sqrt{\alpha - u}} du = \sqrt{2\alpha} \quad \text{for } \alpha > 0, \quad (3.50)$$

$$T_\infty(\alpha) \equiv \frac{1}{\sqrt{2}} \int_0^\alpha \frac{1}{\sqrt{e^\alpha - e^u}} du = \frac{1}{\sqrt{2e^\alpha}} \ln \left( 2e^\alpha - 1 + 2\sqrt{e^\alpha(e^\alpha - 1)} \right) \quad \text{for } \alpha > 0, \quad (3.51)$$

respectively, see (3.1) and (1.6). The following Lemma 3.15(i) determines the shape of  $T_\infty(\alpha)$  on  $(0, \infty)$ , and Lemma 3.15(ii) is a basic comparison theorem for the time map formula  $T_0$ ,  $T_a$  and  $T_\infty$ . Lemma 3.15(i) is obvious, cf. Theorem 1.3. In addition, for fixed  $u > 0$ , since  $\exp\left(\frac{au}{a+u}\right)$  is a strictly increasing function of  $a > 0$ , we obtain that

$$f_0(u) = 1 < f_a(u) = \exp\left(\frac{au}{a+u}\right) = \exp\left(\frac{u}{1+\frac{u}{a}}\right) < \exp u = f_\infty(u) \quad \text{for } a > 0.$$

Thus Lemma 3.15(ii) follows by modification of the proofs of [17, Theorems 2.3–2.4]; we omit the proof.

**Lemma 3.15.** Consider (3.1), (3.50) and (3.51). Let  $\lambda_\infty \approx 0.878$  and  $\alpha_\infty = \ln\left(\frac{2+\lambda_\infty}{\lambda_\infty}\right) \approx 1.187$  be two numbers defined in (1.7) and (1.8), respectively. The following assertions (i)–(ii) hold:

(i)  $\lim_{\alpha \rightarrow 0^+} T_\infty(\alpha) = \lim_{\alpha \rightarrow \infty} T_\infty(\alpha) = 0$ . In addition,  $T'_\infty(\alpha) > 0$  on  $(0, \alpha_\infty)$ ,  $T'_\infty(\alpha_\infty) = 0$  and  $T'_\infty(\alpha) < 0$  on  $(\alpha_\infty, \infty)$ .

(ii) For any fixed  $\alpha > 0$  such that  $T_a(\alpha)$  is a continuous, strictly decreasing function of  $a > 0$ . Moreover,

$$\sqrt{2\alpha} = T_0(\alpha) = \lim_{a \rightarrow 0^+} T_a(\alpha) > T_a(\alpha) > \lim_{a \rightarrow \infty} T_a(\alpha) = T_\infty(\alpha) \quad \text{for } \alpha > 0 \text{ and } a > 0.$$

Throughout this paper, for  $a > a_0$ , by Lemma 3.2, let  $\alpha_M(a)$  and  $\alpha_m(a)$  denote the local maximum and the local minimum points of  $T_a(\alpha)$  on  $(0, \infty)$  where  $\alpha_M < \alpha_m$ , respectively. See Figure 3.1(i).

**Lemma 3.16.** Consider (1.1) with  $a > a_0$ . Then the following assertions (i)–(iii) hold:

(i)  $\alpha_M(a)$  is a strictly decreasing and continuous function of  $a > a_0$ . Furthermore,  $\alpha_M(a) < \omega(a)$  for  $a > a_0$ .

(ii)  $\alpha_m(a)$  is a continuous function of  $a > a_0$ . Furthermore,  $\alpha_m(a)$  is a strictly increasing function on  $(a_0, \tilde{a}]$  and  $p_2(a) \leq \alpha_m(a)$  for  $a \geq \tilde{a}$ .

(iii) For  $a > a_0$ ,

$$\alpha_\infty = \lim_{a \rightarrow \infty} \alpha_M(a) < \alpha_M(a) < \lim_{a \rightarrow a_0^+} \alpha_M(a) = \lim_{a \rightarrow a_0^+} \alpha_m(a) < \alpha_m(a) < \lim_{a \rightarrow \infty} \alpha_m(a) = \infty.$$

**Proof of Lemma 3.16.** We divide this proof into next Steps 1–6.

**Step 1.** We prove that  $\alpha_M(a) < \omega(a)$  for  $a > a_0$ , and  $\alpha_M(a)$  is a strictly decreasing function of  $a$  both on  $(a_0, 6)$  and  $[6, \infty)$ . Let  $a_1 > a_0$  be given. Since  $T'_{a_1}(\alpha_M(a_1)) = 0$  and  $T''_{a_1}(\alpha_M(a_1)) \leq 0$ , and by Lemmas 3.7, 3.8 and 3.12, we observe that  $\alpha_M(a_1) < \eta(a_1) < \omega(a_1)$ . Then we see that  $\alpha_M(a_1) < \omega(a_1) = \omega(a_2)$  for either  $a_0 < a_1 < a_2 < 6$  or  $6 \leq a_1 < a_2$ . So by Lemma 3.12, we see that  $T'_{a_2}(\alpha_M(a_1)) < T'_{a_1}(\alpha_M(a_1)) = 0$  for either  $a_0 < a_1 < a_2 < 6$  or  $6 \leq a_1 < a_2$ . So by Lemma 3.14,  $\alpha_M(a_2) < \alpha_M(a_1)$  for either  $a_0 < a_1 < a_2 < 6$  or  $6 \leq a_1 < a_2$ . Thus,  $\alpha_M(a)$  is a strictly decreasing function both on  $(a_0, 6)$  and  $[6, \infty)$ .

**Step 2.** We prove that  $\alpha_m(a)$  is a strictly increasing function of  $a$  on  $(a_0, \tilde{a}]$ . By Lemma 3.4(i), we see that  $\theta_a(\alpha) > \theta_a(u)$  for  $0 < u < \alpha$  and  $\alpha > p_3(a)$ . It follows that  $T'_a(\alpha) > 0$  for  $\alpha > p_3(a)$  by (3.27). So by Lemma 3.4(i), we further see that  $\alpha_m(a) \leq p_3(a) < 12 = \omega(a)$  for  $a \in (a_0, \tilde{a}]$ . Assume that  $a_0 < a_1 < a_2 \leq \tilde{a}$ . By Lemma 3.12, we observe that  $T'_{a_2}(\alpha_m(a_1)) < T'_{a_1}(\alpha_m(a_1)) = 0$ . It follows that  $\alpha_m(a_1) < \alpha_m(a_2)$  by Lemma 3.14. Thus,  $\alpha_m(a)$  is a strictly increasing function on  $(a_0, \tilde{a}]$ .

**Step 3.** We prove that  $\lim_{a \rightarrow a_0^+} \alpha_M(a) \leq \lim_{a \rightarrow a_0^+} \alpha_m(a)$  and

$$\alpha_\infty = \lim_{a \rightarrow \infty} \alpha_M(a) < \alpha_M(a) < \lim_{a \rightarrow a_0^+} \alpha_M(a) \leq \alpha_m(a) \text{ for } a > a_0. \quad (3.52)$$

Since  $0 < \alpha_M(a) < \omega(a) \leq 12$  for  $a > a_0$ , and by Step 1, we see that  $\lim_{a \rightarrow a_0^+} \alpha_M(a)$  and  $\lim_{a \rightarrow \infty} \alpha_M(a)$  both exist. Suppose to the contrary that  $\lim_{a \rightarrow \infty} \alpha_M(a) \neq \alpha_\infty$ . Then we have two cases. Case 1:  $\lim_{a \rightarrow \infty} \alpha_M(a) < \alpha_\infty$  and Case 2:  $\lim_{a \rightarrow \infty} \alpha_M(a) > \alpha_\infty$ .

Assume that Case 1 holds. Then  $\alpha_M(a_1) < \alpha_\infty$  for some  $a_1 > 6$  by Step 1. We let

$$\delta \equiv \frac{1}{2} \min \{ \alpha_\infty - \alpha_M(a_1), \alpha_m(a_1) - \alpha_M(a_1) \}.$$

Clearly,  $\delta > 0$ . We let  $\alpha \in (\alpha_M(a_1), \alpha_M(a_1) + \delta)$  be given. Then

$$\alpha_M(a_1) < \alpha < \alpha_M(a_1) + \delta < \alpha_m(a_1),$$

$$\alpha_M(a_1) < \alpha < \alpha_M(a_1) + \delta < \alpha_\infty \ (\approx 1.187) < \omega(a) \text{ for } a > a_1. \quad (3.53)$$

So by Lemma 3.15, we have that  $T'_{a_1}(\alpha) < 0$  and  $T'_\infty(\alpha) > 0$ . Then by (3.53), Lemmas 3.6 and 3.12, we find that

$$0 < T'_\infty(\alpha) = \lim_{a \rightarrow \infty} T'_a(\alpha) < T'_{a_1}(\alpha) < 0,$$

which is a contradiction.

Assume that Case 2 holds. We let  $\beta \in (\alpha_\infty, \lim_{a \rightarrow \infty} \alpha_M(a))$  be given. By Step 1 and Lemma 3.15, we see that  $T'_a(\beta) > 0$  and  $T'_\infty(\beta) < 0$  for  $a > 6$ . So by Lemma 3.6(ii) we find that  $0 > T'_\infty(\beta) = \lim_{a \rightarrow \infty} T'_a(\beta) \geq 0$ , which is a contradiction.

Thus, by above discussions, we obtain that  $\lim_{a \rightarrow \infty} \alpha_M(a) = \alpha_\infty$ . In addition, by Lemma 3.4(i), we see that  $\theta_a(\alpha) > \theta_a(u)$  for  $0 < u < \alpha \leq p_1(a)$  and  $a > 4$ . It follows that  $T'_a(\alpha) > 0$  for  $0 < \alpha \leq p_1(a)$  and  $a > 4$  by (3.27). So  $\alpha_M(a) > p_1(a)$  for  $a > a_0$ . Since  $a_0 < 4.08$  by Lemma 3.2, and by Lemma 3.15 and (3.30), we observe that

$$\lim_{a \rightarrow a_0^+} \alpha_M(a) \geq p_1(a_0) \geq p_1(4.08) = \frac{2652}{625} - \frac{102}{625} \sqrt{51} \ (\approx 3.078) > \omega_6 > \alpha_M(6), \quad (3.54)$$

$$\lim_{a \rightarrow 6^-} \alpha_M(a) \geq p_1(6) = 12 - 6\sqrt{3} \ (\approx 1.608) > \alpha_\infty \ (\approx 1.187). \quad (3.55)$$

Since  $\lim_{a \rightarrow \infty} \alpha_M(a) = \alpha_\infty$ , and by Step 1, (3.54) and (3.55), we further observe that

$$\alpha_\infty = \lim_{a \rightarrow \infty} \alpha_M(a) < \alpha_M(a) < \lim_{a \rightarrow a_0^+} \alpha_M(a) \text{ for } a > a_0.$$

Next, we prove that  $\lim_{a \rightarrow a_0^+} \alpha_M(a) \leq \alpha_m(a)$  for  $a > a_0$ . By Lemmas 3.7 and 3.8, we obtain that  $\alpha_M(a) \leq \eta(a)$  for  $a > a_0$ . So by (3.9) and Lemmas 3.5 and 3.8, we observe that

$$\lim_{a \rightarrow a_0^+} \alpha_M(a) \leq \eta(a_0) = \kappa(a_0) \leq \kappa(\tilde{a}) = p_2(\tilde{a}) < p_2(a) < \alpha_m(a) \text{ for } a \geq \tilde{a}. \quad (3.56)$$

Assume that there exists  $a_2 \in (a_0, \tilde{a})$  such that  $\alpha_m(a_2) < \lim_{a \rightarrow a_0^+} \alpha_M(a)$ . Then there exists  $a_3 \in (a_0, a_2)$  such that  $\alpha_m(a_2) < \alpha_M(a_3)$ . Since  $\alpha_m(a_2) < \alpha_M(a_3) < \omega(a)$  for  $a_3 \leq a \leq a_2$ , and by Lemma 3.12, we observe that

$$0 = T'_{a_2}(\alpha_m(a_2)) < T'_{a_2}(\alpha_M(a_3)) < T'_{a_3}(\alpha_M(a_3)) = 0,$$

which is a contradiction. So by (3.56),  $\lim_{a \rightarrow a_0} \alpha_M(a) \leq \alpha_m(a)$  for  $a > a_0$ . So (3.52) holds. In addition, by Step 2 and (3.52), we see that  $\lim_{a \rightarrow a_0^+} \alpha_m(a)$  exists and  $\lim_{a \rightarrow a_0^+} \alpha_M(a) \leq \lim_{a \rightarrow a_0^+} \alpha_m(a)$ .

**Step 4.** We prove that

$$\alpha_M : (a_0, \infty) \rightarrow \left( \alpha_\infty, \lim_{a \rightarrow a_0^+} \alpha_M(a) \right) \text{ is surjective,} \quad (3.57)$$

$$\alpha_m : (a_0, \infty) \rightarrow \left( \lim_{a \rightarrow a_0^+} \alpha_m(a), \infty \right) \text{ is surjective.} \quad (3.58)$$

The proofs of (3.57) and (3.58) are omitted.

**Step 5.** We prove assertions (i) and (ii). By Step 1 and (3.57), we have that  $\alpha_M(a) < \omega(a)$  for  $a > a_0$ , and  $\alpha_M(a)$  is a strictly decreasing and continuous function of  $a$  both on  $(a_0, 6)$  and  $[6, \infty)$ . To prove assertion (i), it is sufficient to prove that  $\alpha_M(6) = \lim_{a \rightarrow 6^-} \alpha_M(a)$ . By (3.57), we see that  $\alpha_M(6) \geq \lim_{a \rightarrow 6^-} \alpha_M(a)$ . Suppose to the contrary that  $\alpha_M(6) > \lim_{a \rightarrow 6^-} \alpha_M(a)$ . Then there exists  $\delta > 0$  such that  $\alpha_M(6) > \alpha_M(a)$  for  $6 - \delta \leq a < 6$ . Furthermore,

$$\alpha_M(a) < \alpha_M(6) < \omega_6 = 3 < \omega(a) \text{ for } 6 - \delta \leq a < 6. \quad (3.59)$$

By (3.59) and Lemma 3.12, we further see that  $T'_6(\alpha_M(a)) < T'_a(\alpha_M(a)) = 0$  for  $6 - \delta \leq a < 6$ . It follows that  $\alpha_M(6) < \alpha_M(a)$  for  $6 - \delta \leq a < 6$ . It is a contradiction. Thus  $\lim_{a \rightarrow 6^-} \alpha_M(a) = \alpha_M(6)$ . It implies that  $\alpha_M(a)$  is a strictly decreasing and continuous function of  $a$  on  $(a_0, \infty)$ . Thus assertion (i) holds.

By Step 2 and (3.58),  $\alpha_m(a)$  is a continuous function on  $(a_0, \tilde{a}]$ . By Lemmas 3.2, 3.5(ii) and 3.8, we see that

$$\alpha_M(a) < \rho(a) \leq \kappa(a) \leq p_2(a) < \alpha_m(a) \text{ for } a \geq \tilde{a}. \quad (3.60)$$

So we further see that  $T'_a(\alpha)$  has a unique zero  $\alpha_m(a)$  on  $[p_2(a), \infty)$  for  $a \geq \tilde{a}$  and

$$T''_a(\alpha_m(a)) = T''_a(\alpha_m(a)) + \frac{2}{\alpha_m(a)} T'_a(\alpha_m(a)) > 0 \text{ for } a \geq \tilde{a}$$

by Lemma 3.7. Then by the Implicit Function Theorem,  $\alpha_m(a)$  is a continuous function on  $[\tilde{a}, \infty)$ . It follows that  $\alpha_m(a)$  is a continuous function on  $(a_0, \infty)$ . Thus by Step 2 and (3.60) assertion (ii) holds.

**Step 6.** We prove assertion (iii). By (3.52), it is sufficient to prove that

$$\lim_{a \rightarrow a_0^+} \alpha_M(a) = \lim_{a \rightarrow a_0^+} \alpha_m(a) < \alpha_m(a) < \lim_{a \rightarrow \infty} \alpha_m(a) = \infty \text{ for } a > a_0. \quad (3.61)$$

By [12, Theorem 2.2], we see that

$$p_1(a) < \|u_{\lambda^*}\|_{\infty} < \gamma(a) = \frac{a(a-2)}{2} < p_2(a) < \|u_{\lambda^*}\|_{\infty} \text{ for } a \geq a^*, \quad (3.62)$$

where  $a^*$  is defined in (1.3). So we observe that  $\lim_{a \rightarrow \infty} \alpha_m(a) \geq \lim_{a \rightarrow \infty} p_2(a) = \infty$ . It follows that  $\alpha_m(a) < \lim_{a \rightarrow \infty} \alpha_m(a) = \infty$  for  $a > a_0$ .

For the sake of convenience, we let  $\alpha^+ \equiv \lim_{a \rightarrow a_0^+} \alpha_M(a)$  and  $\alpha^- \equiv \lim_{a \rightarrow a_0^+} \alpha_m(a)$ . By Step 3,  $\alpha^+ \leq \alpha^-$ . Suppose to the contrary that  $\alpha^+ < \alpha^-$ . Then we assert that  $T'_{a_0}(\beta) > 0$  for some  $\beta \in (\alpha^+, \alpha^-)$ . Otherwise,  $T'_{a_0}(\alpha) = 0$  for all  $\alpha \in (\alpha^+, \alpha^-)$ . It is a contradiction by Lemmas 3.6, 3.7 and 3.11. By (3.52), we find that

$$\alpha_M(a) < \alpha^+ < \beta < \alpha^- < \alpha_m(a) \text{ for } a_0 < a \leq \tilde{a},$$

which implies that  $T'_a(\beta) < 0 < T'_{a_0}(\beta)$  for  $a_0 < a \leq \tilde{a}$ . It is a contradiction by Lemma 3.6(ii). So  $\alpha^+ = \alpha^-$ . Then (3.61) holds. It implies that assertion (iii) holds.

The proof of Lemma 3.16 is complete. ■

We are in a position to prove Lemma 3.3 by applying Lemmas 3.2, 3.5–3.7, 3.11, 3.14 and 3.16.

**Proof of Lemma 3.3.** Since  $\kappa(a) < 8 < \omega(a)$  for  $4 < a \leq a_0$  by Lemma 3.5(i), we see that  $T'_a(\alpha) > T'_{a_0}(\alpha) \geq 0$  for  $0 < \alpha \leq \kappa(a)$  and  $4 < a < a_0$ . Suppose to the contrary that there exists  $\beta_a > \kappa(a)$  for some  $a \in (4, a_0)$  such that  $T'_a(\beta_a) = 0$ . So by Lemma 3.7,  $T''_a(\beta_a) > 0$ . It implies that  $T_a(\alpha)$  has a local minimum point at  $\beta_a > \kappa(a)$ . It is a contradiction by Lemma 3.2. Thus,  $T'_a(\alpha) > 0$  for  $\alpha > 0$  and  $4 < a < a_0$ . So assertion (i) holds.

Next, we prove assertion (ii) of Lemma 3.3. By Lemma 3.16(iii), we obtain that  $\alpha_M(a) < \alpha_0 < \alpha_m(a)$  for  $a > a_0$  where  $\alpha_0 \equiv \lim_{a \rightarrow a_0^+} \alpha_M(a) = \lim_{a \rightarrow a_0^+} \alpha_m(a)$ . It follows that  $T'_a(\alpha_0) < 0$  for  $a > a_0$ . Moreover,  $T'_{a_0}(\alpha_0) \leq 0$  by Lemma 3.6(ii). By Lemmas 3.2 and 3.14,  $T'_{a_0}(\alpha_0) = 0$  and  $T'_{a_0}(\alpha) > 0$  for  $\alpha \in (0, \infty) \setminus \{\alpha_0\}$ . We assert that  $\gamma(a_0) \leq \alpha_0 < \kappa(a_0)$ . Indeed, if  $\alpha_0 < \gamma(a_0)$ , and by Lemma 3.16(i)–(ii), then there exists  $a > a_0$  such that  $\alpha_M(a) < \alpha_0 < \alpha_m(a) < \gamma(a_0)$ . It is a contradiction by Lemma 3.6(i). So  $\gamma(a_0) \leq \alpha_0$ . Since  $T'_{a_0}(\alpha_0) = T''_{a_0}(\alpha_0)$ , and by Lemma 3.7, we find that  $\alpha_0 < \kappa(a_0)$ . So  $\gamma(a_0) \leq \alpha_0 < \kappa(a_0)$ .

The proof of Lemma 3.3 is complete. ■

#### 4. Proof of The Main Result

**Proof of Theorem 2.1.** As mentioned in Section 3, to prove Theorem 2.1(i), (ii) and (iii), it is sufficient to prove that there exists a number  $a_0 \approx 4.069$  satisfying  $4 < a_0 < \tilde{a} \approx 4.107$  such that parts (M1), (M2) and (M3) hold, respectively. Notice that ordering properties of positive solutions of (1.1) in Theorem 2.1(i) can be obtained easily. We have that part (M1) holds immediately by Lemmas 3.1, 3.2 and 3.16(iii); part (M2) holds immediately by Lemmas 3.1 and 3.3(ii); and part (M3) holds immediately by Lemmas 3.1 and 3.3(i). Furthermore, by numerical simulation, we find that  $a_0 \approx 4.069$ .

The proof of Theorem 2.1 is complete. ■



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